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ON SOLVABILITY OF SOME DIFFERENCE-DISCRETE EQUATIONS

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Abstract. Multidimensional difference equations in a discrete half-space are considered. Using the theory of periodic Riemann problems a general solution and solvability conditions in discrete Lebesgue spaces are obtained. Some statements of boundary value problems of discrete type are given.

Keywords: multidimensional difference-discrete equation, symbol, factorization, periodic Riemann problem.

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1. INTRODUCTION

We consider a general multidimensional difference equation in the discrete half-space $\mathbb{Z}^m_+ = \{\tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_m), \tilde{x}_m > 0\}$ of the following type

$$\sum_{|k|=0}^{\infty} a_k(\tilde{x}) u_d(\tilde{x} + \alpha_k) = v_d(\tilde{x}), \quad \tilde{x} \in \mathbb{Z}^m_+, \tag{1.1}$$

where k is a multi-index, $k = (k_1, \dots, k_m)$, $u_d, a_k, |k| = 0, 1, \dots,$ are functions of a discrete variable $\tilde{x} \in \mathbb{Z}^m$, $\{\alpha_k\}$ is given sequence in $\mathbb{Z}^m_+$, v_d is given function of a discrete variable $\tilde{x}$.

Such equations [5, 8, 12, 14–18] arise in many applied problems, for example in control theory [6] and digital signal processing [1], thus the problem of their solvability is very topical. We choose the space L_2 as an initial functional space, but these equations can be considered in more general spaces L_p. Some results related to the one-dimensional continual case can be found in [19]; this study is based on the Wiener-Hopf technique [10].

The equation (1.1) is distinct from the same equation in the whole discrete space $\mathbb{Z}^m$

$$\sum_{|k|=0}^{\infty} a_k(\tilde{x})u_d(\tilde{x}+\alpha_k) = v_d(\tilde{x}), \quad \tilde{x} \in \mathbb{Z}^m,$$

which for constant coefficients $a_k(\tilde{x}) \equiv a_k$ can be solved formally at least by the discrete Fourier transform [1, 11]

$$\tilde{u}(\xi) \equiv (Fu_d)(\xi) = \sum_{|k|=0}^{\infty} u_d(\tilde{x})e^{-i\tilde{x}\cdot\xi}, \quad \xi \in \mathbb{T}^m. \tag{1.2}$$

(Let us note that we mean the discrete Fourier transform as a multidimensional Fourier series.) Indeed, if we will apply this discrete Fourier transform (1.2) to the equation with constant coefficients

$$\sum_{|k|=0}^{\infty} a_k u_d(\tilde{x}+\alpha_k) = v_d(\tilde{x}), \quad \tilde{x} \in \mathbb{Z}^m,$$

we will obtain the following formula

$$\sigma(\xi)\tilde{u}_d(\xi) = \tilde{v}_d(\xi),$$

where

$$\sigma(\xi) = \sum_{|k|=0}^{\infty} a_k e^{-i\alpha_k\cdot\xi}.$$

This implies the solution formula in the Fourier image

$$\tilde{u}_d(\xi) = \sigma^{-1}(\xi)\tilde{v}_d(\xi),$$

if $\sigma(\xi) \neq 0$ for all $\xi \in \mathbb{T}^m$.

2. DISCRETE SPACES AND DIFFERENCE OPERATORS

We take $l^2 \equiv L_2(\mathbb{Z}^m)$ as an initial functional space and consider it as a space consisting of functions u_d of discrete variables $\tilde{x} \in \mathbb{Z}^m$ with corresponding inner product

$$(u_d, v_d) = \sum_{\tilde{x}\in\mathbb{Z}^m} u_d(\tilde{x})\overline{v_d(\tilde{x})}$$

and norm

$$\|u_d\|_2 = \left(\sum_{\tilde{x}\in\mathbb{Z}^m} |u_d(\tilde{x})|^2\right)^{1/2},$$

and consider a general difference-discrete operator of the type

$$(\mathcal{D}u_d)(\tilde{x}) = \sum_{|k|=0}^{\infty} a_k(\tilde{x}) u_d(\tilde{x} + \alpha_k), \quad \tilde{x} \in \mathbb{Z}^m, \ \{\alpha_k\} \subset \mathbb{Z}^m.$$

It is easily verified that if

$$A \equiv \sup_{\tilde{x} \in \mathbb{Z}^m} \sum_{|k|=0}^{\infty} |a_k(\tilde{x})| < +\infty, \tag{2.1}$$

then the operator $\mathcal{D}$ is a bounded linear operator $L_2(\mathbb{Z}^m) \longrightarrow L_2(\mathbb{Z}^m)$.

To say something on solvability of the general equation (1.1) we need to study an operator from the left hand side (1.1) with constant coefficients and to obtain invertibility conditions for this operator. This general concept is called *a local principle* [7].

The mentioned operator is the following

$$(\mathcal{A}u_d)(\tilde{x}) = \sum_{|k|=0}^{\infty} a_k u_d(\tilde{x} + \alpha_k), \quad \tilde{x} \in \mathbb{Z}^m, \ \{\alpha_k\} \subset \mathbb{Z}^m,$$

and the operator $\mathcal{A} : L_2(\mathbb{Z}^m_+) \longrightarrow L_2(\mathbb{Z}^m_+)$ is a linear bounded operator.

It is well known [7] the equation with such operator

$$\mathcal{A}u_d = v_d \tag{2.2}$$

in the space $L_2(\mathbb{Z}^m_+)$ is equivalent to the equation

$$(\mathcal{A}P_+ + IP_-)U_d = V_d$$

in the space $L_2(\mathbb{Z}^m)$ if $P_\pm$ are projectors on $\mathbb{Z}^m_\pm$ and I is an identity operator. This property permits us to consider a more general equation than (2.2) namely the following equation

$$(\mathcal{A}P_+ + \mathcal{B}P_-)U_d = V_d \tag{2.3}$$

in the space $L_2(\mathbb{Z}^m)$, where

$$(\mathcal{B}u_d)(\tilde{x}) = \sum_{|k|=0}^{\infty} b_k u_d(\tilde{x} + \beta_k), \quad \tilde{x} \in \mathbb{Z}^m, \ \{\beta_k\} \subset \mathbb{Z}^m,$$

and

$$B \equiv \sum_{|k|=0}^{\infty} |b_k| < +\infty.$$

3. PERIODIC RIEMANN BOUNDARY PROBLEM

To apply the discrete Fourier transform (1.2) to the equation (2.3) we need to know what are the operators FP_+, FP_-. It was done in papers [16, 18], and here we will briefly describe these constructions. Let us introduce the following operators which are generated by the periodic analogue of the Hilbert transform, $\xi = (\xi', \xi_m)$,

$$(H_{\xi'}^{per}\tilde{u}_d)(\xi) = \frac{1}{2\pi i} v.p. \int\limits_{-\pi}^{\pi} \cot \frac{\xi_m - \eta_m}{2} \tilde{u}_d(\xi', \eta_m) d\eta_m, \quad \xi' \in \mathbb{T}^{m-1},$$

$$P_{\xi'}^{per} = 1/2(I + H_{\xi'}^{per}), \quad Q_{\xi'}^{per} = 1/2(I - H_{\xi'}^{per}).$$

Lemma 3.1. *We have the following relations:*

$$FP_+ = P_{\xi'}^{per} F, \quad FP_- = Q_{\xi'}^{per} F.$$

Thus the equation (2.3) is equivalent to the following equation

$$(\sigma_{\mathcal{A}}(\xi) P_{\xi'}^{per} + \sigma_{\mathcal{B}}(\xi) Q_{\xi'}^{per}) \tilde{U}_d = \tilde{V}_d,$$

where

$$\sigma_{\mathcal{A}}(\xi) = \sum_{|k|=0}^{\infty} a_k e^{-i\alpha_k \cdot \xi}, \qquad \sigma_{\mathcal{B}}(\xi) = \sum_{|k|=0}^{\infty} b_k e^{-i\beta_k \cdot \xi}.$$

One can rewrite the last equation as a one-dimensional singular integral equation with parameter $\xi' \in \mathbb{T}^{m-1}$

$$\frac{\sigma_{\mathcal{A}}(\xi) + \sigma_{\mathcal{B}}(\xi)}{2} \tilde{U}_d(\xi) + \frac{\sigma_{\mathcal{A}}(\xi) - \sigma_{\mathcal{B}}(\xi)}{2} \cdot \frac{1}{2\pi i} v.p. \int\limits_{-\pi}^{\pi} \cot \frac{\xi_m - \eta_m}{2} \tilde{U}_d(\xi', \eta_m) d\eta_m = \tilde{V}_d(\xi). \tag{3.1}$$

The equation (3.1) is closely related to the so called periodic Riemann problem [18]. Let us denote $\Pi_\pm$ the upper and lower half-strips in a complex plane $\mathbb{C}$,

$$\Pi_\pm = \{z \in \mathbb{C} : z = t + is, t \in [-\pi, \pi], \ \pm s > 0\}.$$

The problem is the following. Finding two functions $\Phi^\pm(t), t \in [-\pi, \pi]$, which admit an analytical continuation into $\Pi_\pm$ and satisfy the linear relation

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t),$$

where $G(t), g(t)$ are given functions on $[-\pi, \pi], G(-\pi) = G(\pi), g(-\pi) = g(\pi)$. For our case the corresponding periodic Riemann problem with a parameter ξ' will be the following

$$P_{\xi'}^{per} \tilde{U}_d = -\sigma_{\mathcal{A}}^{-1}(\xi) \cdot \sigma_{\mathcal{B}}(\xi) Q_{\xi'}^{per} \tilde{U}_d + \sigma_{\mathcal{A}}^{-1}(\xi) \cdot \tilde{V}_d, \tag{3.2}$$

because the operators $P_{\xi'}^{per}, Q_{\xi'}^{per}$ are projectors on subspaces $A(\mathbb{T}^m), B(\mathbb{T}^m)$ from $L_2(\mathbb{T}^m)$ consisting of functions admitting bounded analytical continuation on the last variable ξ_m into $\Pi_\pm$ under almost all $\xi' \in \mathbb{T}^{m-1}$. One can remind

$$L_2(\mathbb{T}^m) = A(\mathbb{T}^m) \oplus B(\mathbb{T}^m), \quad P_{\xi'}^{per} L_2(\mathbb{T}^m) = A(\mathbb{T}^m), \quad Q_{\xi'}^{per} L_2(\mathbb{T}^m) = B(\mathbb{T}^m).$$

4. GENERAL SOLUTION OF THE DISCRETE EQUATION

Let us denote $-\sigma_{\mathcal{A}}^{-1}(\xi) \cdot \sigma_{\mathcal{B}} \equiv \sigma(\xi)$. It was shown in [16, 18] that solvability of the problem (3.2) or equivalently equation (3.1) is defined by the so called index of factorization [2–4, 7, 9, 16, 17]. We will give here the needed definition assuming everywhere that

$$\sigma(\xi) \in C(\mathbb{T}^m), \quad \inf_{\xi \in \mathbb{T}^m} |\sigma(\xi)| > 0. \tag{4.1}$$

This symbol $\sigma(\xi)$ satisfying condition (4.1) we call *an elliptic symbol.*

Let us note that $\sigma(\xi)$ is a periodic function, and its period T may be is less than 2π.

Definition 4.1. Factorization of the elliptic symbol $\sigma(\xi)$ is called its representation in the form

$$\sigma(\xi) = \sigma_+(\xi) \cdot \sigma_-(\xi),$$

where $\sigma_+(\xi) \in A(\mathbb{T}^m), \sigma_-(\xi) \in B(\mathbb{T}^m)$.

Definition 4.2. Index of factorization for the elliptic symbol $\sigma(\xi)$ is called the integer

$$\kappa = \frac{1}{2\pi} \int_{-T}^{T} d \arg \sigma(\cdot, \xi_m).$$

Remark 4.3. Such factorization in a sense of definition 1 can be constructed effectively with the help of singular integrals for the case $\kappa = 0$ only.

Remark 4.4. In the theory of boundary value problems for pseudo differential equations [2, 13] the index of factorization coincides as a rule with one half of the order of the pseudo differential operator. We would like to note this concept has a topological nature (see classical works [3, 4, 7, 9]).

Using the theory of periodic Riemann problems [18] one can prove the following result.

Theorem 4.5. *The equation* (2.3) *is uniquely solvable in the space* $L_2(\mathbb{Z}^m)$ *for an arbitrary right hand side* $V_d \in L_2(\mathbb{Z}^m)$ *iff the index of factorization* κ *vanishes.*

The case $\kappa \in \mathbb{N}$ is more interesting. According to [18] there are many solutions for the equation (2.3). We will write these solutions taking into account that arbitrary constants will really depend on ξ'. To formulate the result we need some notation. We put

$$\Gamma^+(\xi', \xi_m) = P_{\xi'}^{per} \left(e^{-i\eta_m \kappa} \sigma(\xi', \eta_m) \right), \quad \Gamma^-(\xi', \xi_m) = Q_{\xi'}^{per} \left(e^{-i\eta_m \kappa} \sigma(\xi', \eta_m) \right),$$

$$\sigma(\xi', \xi_m) = e^{i\xi_m \kappa} \cdot \sigma_+(\xi) \cdot \sigma_-(\xi),$$

where the factors $\sigma_\pm(\xi)$ are elements of factorization for the symbol $e^{-i\xi_m \kappa} \cdot \sigma(\xi)$. We would like to note that the index of factorization for the symbol $e^{-i\xi_m \kappa} \cdot \sigma(\xi)$ is equal to 0.

Further we denote

$$X^+(\xi', \xi_m) = \exp(\Gamma^+(\xi', \xi_m)), \quad X^-(\xi', \xi_m) = e^{i\xi_m \kappa} \cdot \exp(\Gamma^-(\xi', \xi_m)),$$

and write the solution of the periodic Riemann problem in the form

$$(P^{per}_{\xi'} \tilde{U}_d)(\xi', \xi_m) = X^+(\xi', \xi_m) P^{per}_{\xi'} (\sigma_{\mathcal{A}}^{-1} \sigma_+^{-1} \tilde{V}_d)(\xi', \xi_m) + X^+(\xi', \xi_m) S_\kappa(\xi', \xi_m),$$

$$(Q^{per}_{\xi'} \tilde{U}_d)(\xi', \xi_m) = X^-(\xi', \xi_m) Q^{per}_{\xi'} (\sigma_{\mathcal{A}}^{-1} \sigma_+^{-1} \tilde{V}_d)(\xi', \xi_m) + X^-(\xi', \xi_m) S_\kappa(\xi', \xi_m),$$

where $S_\kappa(\xi', \xi_m)$ is a polynomial of order κ on variable ξ_m and type

$$S_\kappa(\xi', \xi_m) = \sum_{j=0}^{\kappa} c_j(\xi') e^{-ij\xi_m}.$$

Hence we prove the following theorem.

Theorem 4.6. *Let $\kappa \in \mathbb{N}$. Then the periodic Riemann problem* (3.2) *in the spaces $A(\mathbb{T}^m), B(\mathbb{T}^m)$ or equivalently the equation* (3.1) *in the space $L_2(\mathbb{T}^m)$ has κ solutions which can be written in the form*

$$(P^{per}_{\xi'} \tilde{U}_d)(\xi', \xi_m) = X^+(\xi', \xi_m) P^{per}_{\xi'} (\sigma_{\mathcal{A}}^{-1} \sigma_+^{-1} \tilde{V}_d)(\xi', \xi_m) + X^+(\xi', \xi_m) S_\kappa(\xi', \xi_m),$$

$$(Q^{per}_{\xi'} \tilde{U}_d)(\xi', \xi_m) = X^-(\xi', \xi_m) Q^{per}_{\xi'} (\sigma_{\mathcal{A}}^{-1} \sigma_+^{-1} \tilde{V}_d)(\xi', \xi_m) + X^-(\xi', \xi_m) S_\kappa(\xi', \xi_m),$$

$$\begin{aligned} \tilde{U}_d(\xi', \xi_m) &= X^+(\xi', \xi_m) P^{per}_{\xi'} (\sigma_{\mathcal{A}}^{-1} \sigma_+^{-1} \tilde{V}_d)(\xi', \xi_m) \\ &+ X^-(\xi', \xi_m) Q^{per}_{\xi'} (\sigma_{\mathcal{A}}^{-1} \sigma_+^{-1} \tilde{V}_d)(\xi', \xi_m) \\ &+ \left[X^+(\xi', \xi_m) + X^-(\xi', \xi_m)\right] S_\kappa(\xi', \xi_m), \end{aligned}$$

where $c_j(\xi'), j = 0, 1, \ldots, \kappa$, are arbitrary functions from $L_2(\mathbb{T}^{m-1})$.

Theorem (4.6) implies that if we want to have a unique solution in the case $\kappa \in \mathbb{N}$ we need some additional conditions to determine uniquely unknown functions $c_j(\xi'), j = 0, 1, \ldots, \kappa$. This case we will discuss in Section 6.

Corollary 4.7. *Let $\kappa \in \mathbb{N}$, $V_d \equiv 0$. A general solution of equation* (2.3) *has the following form*

$$U_d(\tilde{x}', \tilde{x}_m) = \left[X^+(\xi', \xi_m) + X^-(\xi', \xi_m)\right] S_\kappa(\xi', \xi_m).$$

5. SOLVABILITY CONDITIONS

Here we consider the case $\kappa < 0$. According to our periodic Riemann problem [18] we collect the following arguments. In the paper [18] the authors have established a one-to-one correspondence between the periodic Riemann problem and the classical Riemann problem for a unit circle [3,9]. This implies that our problem with a parameter $\xi' \in \mathbb{T}^{m-1}$ for $\Pi_\pm$

$$\Phi^+(\xi', t) = G(\xi', t)\Phi^-(\xi', t) + g(\xi', t), \quad t \in [-\pi, \pi], \tag{5.1}$$

is equivalent to the analogical Riemann problem for the unit circle $\mathbb{S}^1$ [3, 9]

$$\Psi^+(\xi', \tau) = F(\xi', \tau)\Psi^-(\xi', \tau) + f(\xi', \tau), \quad \tau \in \mathbb{S}^1, \quad \tau = e^{it}. \tag{5.2}$$

5.1. EXPANSION FOR A PERIODIC KERNEL

Since solving the last problem is based on properties of the Cauchy type integral we need to use for negative κ the following expansion for the kernel

$$\frac{1}{\tau - z} = -\sum_{j=1}^{|\kappa|} \frac{\tau^{j-1}}{z^j} + \frac{\tau^{|\kappa|}}{z^{|\kappa|}(\tau - z)}, \quad z \in D^-, \tag{5.3}$$

where D^- is an outer of a unit circle.

Since solutions of problems (5.1),(5.2) are written by a singular integral with Cauchy kernel and kernel $\cot \frac{t}{2}$ respectively we take into account the following correlation between these two integrals. Indeed, if $\xi = e^{i\eta}$ then we have

$$\int_{\mathbb{S}^1} \frac{\varphi(\tau) d\tau}{\tau - \xi} = i \int_{-\pi}^{\pi} \frac{\phi(t) e^{it} dt}{e^{it} - e^{i\eta}} = i \int_{-\pi}^{\pi} \frac{\phi(t)(e^{it} + e^{i\eta}) dt}{e^{it} - e^{i\eta}} - i e^{i\eta} \int_{-\pi}^{\pi} \frac{\phi(t) dt}{e^{it} - e^{i\eta}},$$

where $\phi(t) = \varphi(e^{it})$, and further taking into account

$$i \frac{e^{it} + e^{i\eta}}{e^{it} - e^{i\eta}} = \cot \frac{t - \eta}{2}, \quad \frac{1}{\tau(\tau - \xi)} = -\frac{1}{\xi} \left(\frac{1}{\tau} - \frac{1}{\tau - \xi} \right),$$

we have

$$\int_{\mathbb{S}^1} \frac{\varphi(\tau) d\tau}{\tau - \xi} = \int_{-\pi}^{\pi} \cot \frac{t - \eta}{2} \phi(t) dt + \int_{\mathbb{S}^1} \frac{\varphi(\tau) d\tau}{\tau}.$$

Thus each expansion (5.3) generates the corresponding expansion ($z = e^{i\zeta}$)

$$\begin{aligned} \int_{\mathbb{S}^1} \frac{\varphi(\tau) d\tau}{\tau - z} &= -\sum_{j=1}^{|\kappa|} \int_{\mathbb{S}^1} \frac{\tau^{j-1} \varphi(\tau) d\tau}{z^j} + \int_{\mathbb{S}^1} \frac{\tau^{|\kappa|} \varphi(\tau) d\tau}{z^{|\kappa|}(\tau - z)} \\ &= \sum_{j=0}^{|\kappa|} c_j e^{-ij\zeta} + e^{i\kappa\zeta} \int_{-\pi}^{\pi} \cot \frac{t - \zeta}{2} \phi(t) e^{it|\kappa|} dt, \end{aligned}$$

where

$$c_j = \int_{\mathbb{S}^1} \tau^{j-1} \varphi(\tau) d\tau = i \int_{-\pi}^{\pi} e^{ijt} \phi(t) dt, \quad j = 0, 1, \dots, |\kappa|.$$

Hence we have

$$\int_{-\pi}^{\pi} \cot \frac{t - \zeta}{2} \phi(t) dt = \sum_{j=0}^{|\kappa|} c_j e^{-ij\zeta} + e^{i\kappa\zeta} \int_{-\pi}^{\pi} \cot \frac{t - \zeta}{2} \phi(t) e^{it|\kappa|} dt. \tag{5.4}$$

5.2. REPRESENTATION FOR THE SOLUTION

Here we recall that the periodic Riemann problem really depends on a parameter ξ'. Thus all coefficients c_j in the formula (5.4) will depend on ξ'. We recall solving the algorithm for problem (5.1) taking into account that for our case we have (see formula (3.2))

$$\Phi^+(\xi',\xi_m)=\sigma(\xi',\xi_m)\Phi^-(\xi',\xi_m)+\sigma_{\mathcal{A}}^{-1}(\xi',\xi_m)\tilde{V}_d(\xi',\xi_m).$$

Further we write after factorization

$$\begin{aligned}\sigma_+^{-1}(\xi',\xi_m)\Phi^+(\xi',\xi_m)&=e^{i\kappa\xi_m}\sigma_-(\xi',\xi_m)\Phi^-(\xi',\xi_m)\\&\quad+\sigma_+^{-1}(\xi',\xi_m)\sigma_{\mathcal{A}}^{-1}(\xi',\xi_m)\tilde{V}_d(\xi',\xi_m),\end{aligned}\tag{5.5}$$

decompose the right hand side $\sigma_+^{-1}(\xi',\xi_m)\sigma_{\mathcal{A}}^{-1}(\xi',\xi_m)\tilde{V}_d(\xi',\xi_m)\equiv g(\xi',\xi_m)$ on two summands

$$g(\xi',\xi_m)=g_+(\xi',\xi_m)+g_-(\xi',\xi_m),$$

and write finally the following

$$\sigma_+^{-1}(\xi',\xi_m)\Phi^+(\xi',\xi_m)-g_+(\xi',\xi_m)=g_-(\xi',\xi_m)-e^{i\kappa\xi_m}\sigma_-(\xi',\xi_m)\Phi^-(\xi',\xi_m).\tag{5.6}$$

To explain the equality (5.6) we need a theorem of Liouville type [3, 9], therefore we transfer problem (5.6) to a unit circle when new unknown functions $\Psi^{\pm}(\xi',\tau)$ are boundary values of analytical functions in inner and outer domains of a unit circle $D^{\pm}$

$$\gamma_+^{-1}(\xi',\tau)\Psi^+(\xi',\tau)-h_+(\xi',\tau)=h_-(\xi',\tau)-\tau^{\kappa}\gamma_-(\xi',\tau)\Psi^-(\xi',\tau),\tag{5.7}$$

where $\tau\in\mathbb{S}^1,\tau=e^{i\xi_m}$.

The right hand side of the equality (5.7) has a zero at infinity as a Cauchy type integral so that by the Liouville theorem this is identical zero. Thus,

$$\Psi^+(\xi',\tau)=\gamma_+(\xi',\tau)h_+(\xi',\tau),\quad \Psi^-(\xi',\tau)=\tau^{-\kappa}\gamma_-^{-1}(\xi',\tau)h_-(\xi',\tau),$$

which implies the function Ψ^- has a pole of order $|\kappa|$ at infinity. To exclude such a possibility we need to use the expansion of the type (5.3) for a Cauchy type kernel, or the expansion (5.4) for our periodic kernel.

If we return to the equality (5.6) we obtain

$$\Phi^+(\xi',\xi_m)=\sigma_+(\xi',\xi_m)g_+(\xi',\xi_m),$$

and taking into account (5.4)

$$\begin{aligned}&\Phi^-(\xi',\xi_m)\\&=e^{-i\kappa\xi_m}\sigma_-^{-1}(\xi',\xi_m)g_-(\xi',\xi_m)=\sigma_-^{-1}(\xi',\xi_m)\sum_{j=0}^{|\kappa|}c_j(\xi')e^{-i(j+\kappa)\xi_m}\\&\quad+\sigma_-^{-1}(\xi',\xi_m)\int_{-\pi}^{\pi}\cot\frac{\eta_m-\xi_m}{2}\sigma_+^{-1}(\xi',\eta_m)\sigma_{\mathcal{A}}^{-1}(\xi',\eta_m)\tilde{V}_d(\xi',\eta_m)e^{i\eta_m|\kappa|}d\eta_m,\end{aligned}\tag{5.8}$$

where according to our evaluations

$$c_j(\xi') = \int\limits_{-\pi}^{\pi} \sigma_+^{-1}(\xi', \eta_m)\sigma_{\mathcal{A}}^{-1}(\xi', \eta_m)\tilde{V}_d(\xi', \eta_m)e^{i\eta_m j}d\eta_m, \quad j = 0, 1, \dots, |\kappa|. \tag{5.9}$$

In the formula for Φ^- one can see that $|\kappa|$ summands have an exponential growth at infinity in the domain Π_-. Indeed, for $\zeta = \xi_m - i\tau, \tau > 0$, we have

$$e^{-i(j+\kappa)\zeta} = e^{-i(j+\kappa)(\xi_m - i\tau)} = e^{-i(j+\kappa)\xi_m}e^{-(j+\kappa)\tau},$$

and since $j + \kappa < 0$ for $j = 0, 1, \dots, |\kappa| - 1$, we have an exponential growth at infinity.

We can collect our considerations of this section in the following theorem.

Theorem 5.1. *Let $\kappa < 0$ be an index of factorization of $\sigma(\xi', \xi_m)$. Then the equation (2.3) has a solution in the space $L_2(\mathbb{Z}^m)$ iff $c_j(\xi') = 0$ for all $\xi' \in \mathbb{T}^{m-1}$, $j = 0, 1 \dots, |\kappa| - 1$.*

Conditions (5.9) might be written in the initial space $L_2(\mathbb{Z}^m)$. Let $\tilde{\Delta}_j : L_2(\mathbb{T}^m) \to L_2(\mathbb{T}^m)$ be an operator of type

$$\tilde{\Delta}_{j1} : \ \tilde{u}_d(\xi) \longmapsto e^{i\xi_j}\tilde{u}_d(\xi), \quad \xi \in \mathbb{T}^m.$$

Obviously it corresponds to the operator $\Delta_j : L_2(\mathbb{Z}^m) \to L_2(\mathbb{Z}^m)$ of the type

$$\Delta_{j1} : u_d(\tilde{x}) \longmapsto u_d(\tilde{x}_1, \dots, \tilde{x}_j + 1, \dots, \tilde{x}_m), \quad \tilde{x} \in \mathbb{Z}^m.$$

In other words the operator Δ_j is a difference operator of first order on $\tilde{x}_j$, more precisely it is a right shift operator on variable $\tilde{x}_j$.

Thus the operator

$$\tilde{\Delta}_{mj} : \tilde{u}_d(\xi) \longmapsto e^{ij\xi_m}\tilde{u}_d(\xi), \quad \xi \in \mathbb{T}^m,$$

corresponds to the operator

$$\Delta_{mj} : u_d(\tilde{x}) \longmapsto u_d(\tilde{x}_1, \dots, \tilde{x}_m + j), \quad \tilde{x} \in \mathbb{Z}^m.$$

We recall one property of the discrete Fourier transform related to restriction on a discrete hyper-plane. We consider a restriction of the function $u_d(\tilde{x})$ on the discrete hyper-plane $\tilde{x}_m = 0$, i.e. $\mathbb{Z}^{m-1}$. According to the inverse Fourier transform we have

$$u_d(\tilde{x}', \tilde{x}_m) = \frac{1}{(2\pi)^m}\int\limits_{\mathbb{T}^m} e^{i\tilde{x}'\cdot\xi'}e^{i\tilde{x}_m\cdot\xi_m}\tilde{u}_d(\xi', \xi_m)d\xi' d\xi_m,$$

hence

$$\begin{aligned} u_d(\tilde{x}', 0) &= \frac{1}{(2\pi)^m}\int\limits_{\mathbb{T}^m} e^{i\tilde{x}'\cdot\xi'}\tilde{u}_d(\xi', \xi_m)d\xi' d\xi_m \\ &= \frac{1}{(2\pi)^m}\int\limits_{\mathbb{T}^{m-1}} e^{i\tilde{x}'\cdot\xi'}\left(\int\limits_{-\pi}^{\pi}\tilde{u}_d(\xi', \xi_m)d\xi_m\right)d\xi', \end{aligned}$$

and we see that restriction on a hyper-plane corresponds to integration of the Fourier image on the last variable. Taking into account this fact and recalling that multiplication in Fourier images corresponds to a convolution operator in the original discrete space $L_2(\mathbb{Z}^m)$ we can write the following condition instead of the (5.9)

$$(\Delta_{mj}C_jV_d)(\tilde{x}',0)=0 \quad \text{for all} \quad \tilde{x}'\in\mathbb{Z}^{m-1}, \quad j=0,1,\dots,|\kappa|,$$

where C_j is a discrete convolution operator with the symbol

$$\sigma_+^{-1}(\xi',\xi_m)\sigma_{\mathcal{A}}^{-1}(\xi',\xi_m)e^{i\xi_m j}.$$

Remark 5.2. One can prove that representation (5.8) is applicable and useful if we will enlarge the spaces of possible solutions (cf. [2]). Maybe such spaces will be like spaces $H_p(\Gamma)$ which were introduced by V.S. Vladimirov for the spaces of analytical functions in a radial tube domain over the cone Γ [20].

6. BOUNDARY VALUE PROBLEMS IN DISCRETE SPACES

This section is a direct continuation of Section 4 and gives a statement of a simple boundary value problem for equation (2.3). We start from a formula for the general solution for equation (2.3) including unknown functions $c_j(\xi'), j=0,1,\dots,\kappa,\kappa\in\mathbb{N}$. For simplicity we consider a homogeneous equation (2.3) although all results will be valid for the inhomogeneous case without additional special requirements.

Let us introduce the following boundary conditions:

$$(B_ju_d)(\tilde{x}',0)=b_j(\tilde{x}'), \quad j=0,1,\dots,\kappa, \tag{6.1}$$

where B_j can be a discrete convolution operator with kernel $B_j(\tilde{x})$ and symbol $\tilde{B}_j(\xi)\in L_\infty(\mathbb{T}^m)$

$$B_j: u_d(\tilde{x})\longmapsto\sum_{\tilde{y}\in\mathbb{Z}^m}B_j(\tilde{x}-\tilde{y})u_d(\tilde{y})$$

or it may be a difference-discrete operator of the type

$$B_j: u_d(\tilde{x})\longmapsto\sum_{|k|=0}^{l}r_ku_d(\tilde{x}+\gamma_k), \quad \{\gamma_k\}\subset\mathbb{Z}^m,$$

such that its symbol $\tilde{B}_j(\xi)$ is a bounded function.

If we will rewrite boundary conditions (6.1) in Fourier images

$$\int_{-\pi}^{\pi}\tilde{B}_j(\xi',\xi_m)\tilde{u}_d(\xi',\xi_m)d\xi_m=\tilde{b}_j(\xi'), \quad j=0,1,\dots,\kappa, \tag{6.1'}$$

then we can prove the following result.

Theorem 6.1. *If $\kappa \in \mathbb{N}$ then the boundary value problem* (2.3), (6.1) *has a unique solution in the space $L_2(\mathbb{Z}^m)$ iff*

$$\det(s_{kj}(\xi'))_{k,j=0}^{\kappa} \neq 0 \quad \textit{for all} \quad \xi' \in \mathbb{T}^{m-1}. \tag{6.2}$$

Proof. Substituting the general solution of the equation (2.3) into boundary conditions (6.1)' we have

$$\int\limits_{-\pi}^{\pi} \tilde{B}_j(\xi', \xi_m) X(\xi', \xi_m) \sum_{k=0}^{\kappa} c_k(\xi') e^{ik\xi_m} d\xi_m = \tilde{b}_j(\xi'), \quad j = 0, 1, \dots, \kappa,$$

where $X(\xi', \xi_m) = X^+(\xi', \xi_m) + X^-(\xi', \xi_m)$, and further

$$\sum_{k=0}^{\kappa} c_k(\xi') \int\limits_{-\pi}^{\pi} \tilde{B}_j(\xi', \xi_m) X(\xi', \xi_m) e^{ik\xi_m} d\xi_m = \tilde{b}_j(\xi'), \quad j = 0, 1, \dots, \kappa.$$

Renaming

$$s_{jk}(\xi') = \int\limits_{-\pi}^{\pi} \tilde{B}_j(\xi', \xi_m) X(\xi', \xi_m) e^{ik\xi_m} d\xi_m,$$

we obtain the following system of linear algebraic equations

$$\sum_{k=0}^{\kappa} s_{jk}(\xi') c_k(\xi') = \tilde{b}_j(\xi'), \quad j = 0, 1, \dots, \kappa, \tag{6.3}$$

with respect to unknown functions $c_k(\xi'), k = 0, 1, \dots, \kappa$. The condition (6.2) is necessary and sufficient for unique solvability of the inhomogeneous system (6.3). $\square$

Condition (6.2) is a variant of Shapiro–Lopatinskii condition [2].

7. COBOUNDARY OPERATORS AND UNKNOWN POTENTIALS

In this section everywhere $-\kappa \in \mathbb{N}$. Collecting our evaluations from Section 5.2 we can formulate the following lemma.

Lemma 7.1. *There exists a unique collection of functions $c_j(\xi') \in L_2(\mathbb{T}^{m-1})$, $j = 0, 1, \dots, |\kappa|$, such that the following representation*

$$\int\limits_{-\pi}^{\pi} \cot \frac{\eta_m - \xi_m}{2} g(\xi', \xi_m) d\eta_m$$
$$= \sum_{j=0}^{|\kappa|} c_j(\xi') e^{-ij\xi_m} + e^{i\kappa\xi_m} \int\limits_{-\pi}^{\pi} \cot \frac{\eta_m - \xi_m}{2} g(\xi', \xi_m) e^{i\eta_m |\kappa|} d\eta_m,$$

where

$$c_j(\xi') = i\int\limits_{-\pi}^{\pi} e^{ij\xi_m} g(\xi',\xi_m)d\xi_m, \quad j = 0,1,\dots,|\kappa|,$$

holds for all $g(\xi',\xi_m) \in L_2(\mathbb{T}^m)$.

Now we will return to formulas (5.6), (5.8):

$$\Phi^+(\xi',\xi_m) = \sigma_+(\xi',\xi_m)g_+(\xi',\xi_m),$$

$$\Phi^-(\xi',\xi_m) = \sigma_-^{-1}(\xi',\xi_m)\sum_{j=0}^{|\kappa|} c_j(\xi')e^{-i(j+\kappa)\xi_m}$$

$$+\sigma_-^{-1}(\xi',\xi_m)\int\limits_{-\pi}^{\pi} \cot\frac{\eta_m-\xi_m}{2} g(\xi',\eta_m)e^{i\eta_m|\kappa|}d\eta_m,$$

If in the statement of the periodic Riemann problem (3.2) we will admit that the space $B(\mathbb{T}^m)$ might include some functions of exponential growth then the last formula for Φ^- is applicable and we have $|\kappa|+1$ arbitrary functions. In other words the problem is over-determined and we can introduce additional unknown functions c_j in the statement of the periodic Riemann problem (3.2) (instead of solvability conditions for the right hand side). If so we will formulate a more general equation than equation (2.3).

Evidently if we define the Fourier transform $F : L_2(\mathbb{Z}^m) \to L_2(\mathbb{T}^m)$ by the formula

$$\tilde{u}_d(\xi) = (Fu_d)(\xi) = \sum_{\tilde{x}\in\mathbb{Z}^m} e^{-i\tilde{x}\cdot\xi}u_d(\tilde{x}),$$

then the inverse Fourier transform $F^{-1} : L_2(\mathbb{T}^m) \to L_2(\mathbb{Z}^m)$ should be

$$u_d(\tilde{x}) = (F^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^m}\int\limits_{\mathbb{T}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}_d(\xi)d\xi.$$

In other words numbers $u_d(\tilde{x})$ are Fourier coefficients of the function $\tilde{u}_d(\xi)$. Now we will define an discrete analogue of a one-dimensional discrete indicator function by the following way. We put $\delta(\tilde{x}_m)$ as

$$\delta(\tilde{x}_m) = \begin{cases} 1, & \text{if } \tilde{x}_m = 0, \\ 0, & \text{in other cases.} \end{cases}$$

So the one-dimensional discrete Fourier transform of such a function is

$$(F\delta)(\xi_m) = 1.$$

Thus in the case $-\kappa \in \mathbb{N}$ we consider the equation

$$(\mathcal{A}P_+ + \mathcal{B}P_-)U_d + \sum_{j=0}^{|\kappa|} \mathcal{K}_j\left(c_j(\tilde{x}') \otimes \delta(\tilde{x}_m)\right) = V_d, \tag{7.1}$$

where we have unknowns $U_d, c_j, j = 0, 1, \ldots, |\kappa|$, and $\mathcal{K}_j$ is a pseudo differential operator with the symbol $K_j(\xi', \xi_m) \in L_\infty(\mathbb{T}^m)$.

Theorem 7.2. *Let $-\kappa \in \mathbb{N}$. Then the equation* (7.1) *has unique solution $U_d \in L_2(\mathbb{Z}^m)$, $c_j \in L_2(\mathbb{Z}^{m-1}), j = 0, 1, \ldots, |\kappa|$, iff*

$$\det(t_{kj}(\xi'))_{k,j=0}^{|\kappa|} \neq 0 \quad \textit{for all} \quad \xi' \in \mathbb{T}^{m-1}. \tag{7.2}$$

Proof. We apply the Fourier transform to the equation (7.1) and reduce it to the equation (see the formula (3.2))

$$P_{\xi'}^{per}\tilde{U}_d = -\sigma_{\mathcal{A}}^{-1}(\xi) \cdot \sigma_{\mathcal{B}}(\xi) Q_{\xi'}^{per}\tilde{U}_d + \sigma_{\mathcal{A}}^{-1}(\xi) \cdot \tilde{V}_d - \sigma_{\mathcal{A}}^{-1}(\xi) \sum_{j=0}^{|\kappa|} K_j(\xi', \xi_m)\tilde{c}_j(\xi')e^{ij\xi_m},$$

in other words the right hand side of our periodic Riemann problem is

$$\sigma_{\mathcal{A}}^{-1}(\xi) \cdot \left(\tilde{V}_d(\xi) - \sum_{j=0}^{|\kappa|} K_j(\xi', \xi_m)\tilde{c}_j(\xi')e^{ij\xi_m}\right).$$

Further we need solvability conditions (5.9) from which we obtain

$$\int_{-\pi}^{\pi} \sigma_+^{-1}(\xi', \eta_m)\sigma_{\mathcal{A}}^{-1}(\xi', \eta_m) \left(\tilde{V}_d(\xi', \eta_m) - \sum_{n=0}^{|\kappa|} K_j(\xi', \eta_m)\tilde{c}_n(\xi')\right) e^{in\eta_m} d\eta_m = 0,$$

$$j = 0, 1, \ldots, |\kappa|.$$

It is left to introduce new notations

$$\int_{-\pi}^{\pi} \sigma_+^{-1}(\xi', \eta_m)\sigma_{\mathcal{A}}^{-1}(\xi', \eta_m)\tilde{V}_d(\xi', \eta_m)e^{in\eta_m} d\eta_m = f_n(\xi'),$$

$$t_{jn}(\xi') = \int_{-\pi}^{\pi} \sigma_+^{-1}(\xi', \eta_m)\sigma_{\mathcal{A}}^{-1}(\xi', \eta_m)K_j(\xi', \eta_m)\tilde{c}_n(\xi')e^{in\eta_m} d\eta_m,$$

so that we have a inhomogeneous system of linear algebraic equations with respect to unknowns $\tilde{c}_n(\xi'), n = 0, 1, \ldots, |\kappa|$, with the matrix $(t_{kj}(\xi'))_{k,j=0}^{|\kappa|}$. The condition (7.2) is necessary and sufficient for this system to be uniquely solvable. □

CONCLUSION

There are a lot of unconsidered problems in this paper. So, for example these results can be transferred on a continual case with some corrections. This will be the subject of another paper.

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