TRANSITION RADIATION BY RELATIVISTIC ELECTRONS IN INHOMOGENEOUS SUBSTANCE

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Abstract: The process of transition radiation of relativistic electrons in non-uniform media is considered. The method of description of this process based on the equivalent photons method and the eikonal approximation of the wave mechanics is proposed. The formulae for the spectral-angular density of the transition radiation that permit to examine the radiation in the case when the dielectric permittivity depends on more than one coordinate are obtained in this approximation. The comparison of the basic results obtained in Born and eikonal approximations of the transition radiation theory is carried out. The ranges of validity of these results are determined. The formulae obtained are applied to the analysis of the transition radiation process on the uniform plate and on the fiber-like target.

Keywords: transition radiation, Born approximation, eikonal approximation, fiber-like target, nanotube

1. INTRODUCTION

Transition radiation (TR) arises when a charged particle crosses the boundary between two media with different dielectric properties [1-3]. Commonly the description of this process is carried out via “sewing” the fields generated by the particle in substances on their boundary. However, such approach to TR description could be developed only for the media with the simplest shape of the boundary between them (plane, spherical or cylindrical boundary [1-4]). In addition, it is assumed commonly that the dielectric permittivity of each medium is constant. For the cases of complicated boundary configuration and fuzzy boundaries another approaches to the TR process description are to develop.

One of such approaches to description of TR of ultrarelativistic particles in the range of high frequencies is method based on the expansion of the radiation amplitude by small deviation of the dielectric permittivity from the unit, analogous to Born expansion in quantum theory of scattering [2,3]. However, the condition of validity of that expansion rapidly violates with the radiated photon frequency decrease. So, the development of methods that permit to work out of the frames of Born perturbation theory in the problem under consideration is necessary.

In the present paper the possibility of use of the eikonal approximation to description of the process of transition radiation by relativistic electrons in the medium with non-uniform dielectric permittivity is studied. The approximated method of TR description is based on the presentation of the particle's field in the form of the packet of free electromagnetic waves and application of the eikonal approximation for the description of the scattering of this wave packet on non-uniformities of the dielectric permittivity of the medium.

2. SPECTRAL-ANGULAR DENSITY OF TR

Consider the particle with electric charge e moving with the constant velocity $\vec{v}$ in non-uniform medium with dielectric permittivity $\varepsilon_\omega(\vec{r})$. In this case Fourier component of the electric field

$$\vec{E}_\omega(\vec{r}) = \int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, e^{i\omega t} dt$$

created in the target under passage of the particle satisfies the equations [2]

$$\left(\Delta + \omega^2 \varepsilon_\omega\right)\vec{E}_\omega = grad\, div\, \vec{E}_\omega - 4\pi e i\omega \frac{\vec{v}}{v}\delta(\vec{\rho}) e^{i\omega \frac{z}{v}}, \qquad (2.1)$$

$$div\, \varepsilon_\omega \vec{E}_\omega(\vec{r}) = 4\pi e \delta(\vec{\rho}) e^{i\omega \frac{z}{v}}. \qquad (2.2)$$

For $\varepsilon_\omega = 1$ the solution of Eqs. (2.1), (2.2) is Coulomb field of the particle

$$\vec{E}_\omega^{(C)}(\vec{r}) = \frac{2e\omega}{v^2\gamma} e^{i\frac{\omega}{v}z} \left\{ \frac{\vec{\rho}}{\rho} K_1\left(\frac{\omega\rho}{v\gamma}\right) - i\frac{\vec{v}}{v}\frac{1}{\gamma} K_0\left(\frac{\omega\rho}{v\gamma}\right) \right\} =$$

$$= \frac{ie}{\pi v} e^{i\frac{\omega}{v}z} \int \frac{\vec{\kappa} + \frac{\omega}{v^2\gamma^2}\vec{v}}{\omega^2 - \frac{\omega^2}{v^2} - \vec{\kappa}^2} e^{i\vec{\kappa}\vec{\rho}} d^2\kappa, \quad \vec{\kappa} \perp \vec{v}, \tag{2.3}$$

where $K_n(x)$ is the modified Bessel function of the third kind, $\gamma = (1 - v^2)^{-1/2}$ is Lorentz-factor of the particle. Using (2.2) and (2.3) we can transform Eq. (2.1) to the form

$$\left(\Delta + \omega^2\right)\left(\vec{E}_\omega - \vec{E}_\omega^{(C)}\right) = \omega^2 (1 - \varepsilon_\omega)\vec{E}_\omega + grad\, div\left[(1 - \varepsilon_\omega)\vec{E}_\omega\right]. \tag{2.4}$$

The last equation could be written in the integral form:

$$\vec{E}_\omega - \vec{E}_\omega^{(C)} = \int d^3r'\, G(\vec{r} - \vec{r}\,') \times$$

$$\times \left\{\omega^2 (1 - \varepsilon_\omega(\vec{r}\,'))\vec{E}_\omega(\vec{r}\,') + grad\, div\left[(1 - \varepsilon_\omega(\vec{r}\,'))\vec{E}_\omega(\vec{r}\,')\right]\right\}, \tag{2.5}$$

where $G(\vec{r} - \vec{r}\,')$ is Green function for Eq. (2.4),

$$G(\vec{r} - \vec{r}\,') = \int \frac{e^{i\vec{\kappa}(\vec{r} - \vec{r}\,')}}{\omega^2 - \vec{\kappa}^2 + i0} \frac{d^3\kappa}{(2\pi)^3}, \tag{2.6}$$

where $\vec{k} = \omega\vec{r}/r$. To find the field of radiation, we need to use the asymptotic of (2.6) on large distances from the region where $\varepsilon_\omega(\vec{r})$ is not equal to unit:

$$G(\vec{r} - \vec{r}\,')\Big|_{r\to\infty} \to -\frac{1}{4\pi}\frac{e^{i\omega r}}{r} e^{-i\vec{k}\vec{r}\,'}. \tag{2.7}$$

Substituting it into (2.5) we obtain the following expression for the radiation field:

$$\vec{E}_\omega^{(rad)}(\vec{r}) = \left(\vec{E}_\omega - \vec{E}_\omega^{(C)}\right)\Big|_{r\to\infty} = -\frac{1}{4\pi}\frac{e^{i\omega r}}{r}\left(\omega^2\vec{I} - \vec{k}(\vec{k}\cdot\vec{I})\right), \tag{2.8}$$

where

$$\vec{I} = \int d^3r\, e^{-i\vec{k}\vec{r}} (1 - \varepsilon_\omega(\vec{r}))\vec{E}_\omega(\vec{r}). \tag{2.9}$$

Computing the flux of Pointing vector into the solid angle element do far from the target (see [6], Eq. (66.9)) we find that the radiated wave intensity is equal to

$$\frac{d\mathrm{E}}{d\omega do} = r^2 \frac{1}{4\pi^2} \left| \vec{E}_\omega^{(rad)}(\vec{r}) \right|^2 .$$

Substituting (2.7) into the last formula we obtain the following expression for the spectral-angular density of TR:

$$\frac{d\mathrm{E}}{d\omega do} = \frac{\omega^2}{(8\pi^2)^2} \left| \vec{k} \times \vec{I} \right|^2 , \tag{2.10}$$

where ω and $\vec{k}$ are the frequency and wave vector of the radiated wave.

Consider TR in the range of high frequencies, where the dielectric permittivity of the target is determined by the formula

$$\varepsilon_\omega(\vec{r}) \approx 1 - \frac{\omega_p^2}{\omega^2}, \quad \omega >> \omega_p , \tag{2.11}$$

where $\omega_p = \sqrt{4\pi e^2 n(\vec{r})/m}$ is the plasma frequency, m and e are the charge and mass of an electron, $n(\vec{r})$ is the electron density in the target. In this case the solution of Eqs. (2.1), (2.2) could be found as an expansion by the small value $(1-\varepsilon_\omega)$. In the first order of such expansion (that corresponds to Born approximation) the solution of Eqs. (2.1), (2.2) is Coulomb field of the particle (2.3). So the substitution $\vec{E}_\omega(\vec{r}) = \vec{E}_\omega^{(C)}(\vec{r})$ into (2.9) corresponds to Born approximation in TR theory. It is easy to see that characteristic values of the transverse (perpendicular to $\vec{v}$) component of the Coulomb field of relativistic particle exceed the characteristic values of the longitudinal component in γ times. So, neglecting the terms of the order of γ^{-2}, we can hold on in (2.10) only transverse component of the vector $\vec{I}$ ($\vec{I} \approx \vec{I}_\perp$). In this case

$$\frac{d\mathrm{E}}{d\omega do} = \frac{\omega^4}{(8\pi^2)^2} \left| \vec{I}_\perp^{(B)} \right|^2 ,$$

$$\vec{I}_\perp^{(B)} = \int d^3 r\, e^{-i\vec{k}\vec{r}} (1 - \varepsilon_\omega(\vec{r})) \vec{E}_\omega^{(C)}(\vec{r})_\perp . \tag{2.12}$$

Keeping in mind the following comparison of the results of Born and eikonal approximations in TR theory and their ranges of validity, consider the simplest problem on TR under normal incidence of the particle to the uniform plate of the thickness a. Born approximation leads us to the following result in this case:

$$\vec{I}_{\perp}^{(B)} = -\frac{4\pi e}{v}\frac{\omega_p^2/\omega^2}{\frac{\omega}{v}-k_z}\frac{\vec{k}_{\perp}}{k_{\perp}^2+\left(\frac{\omega}{v\gamma}\right)^2}\left\{\exp\left[i\left(\frac{\omega}{v}-k_z\right)a\right]-1\right\}, \qquad (2.13)$$

$$\frac{d\mathrm{E}}{d\omega do} = 2\frac{e^2}{\pi^2}\left(\frac{\omega_p^2}{\omega^2}\right)^2\frac{\theta^2}{\left(\theta^2+\gamma^{-2}\right)^4}\left\{1-\cos\left[\left(\theta^2+\gamma^{-2}\right)\frac{\omega a}{2}\right]\right\}. \qquad (2.14)$$

3. TR ON A FIBER-LIKE TARGET IN BORN APPROXIMATION

Born approximation in TR theory permits to consider the radiation on the targets of rather complicated configuration. Particularly, TR on fiber-like targets and nanotubes was considered [7,8] using Born approximation. Let us briefly remember that results.

Consider transition radiation of a relativistic particle incident on a thin dielectric fiber at small angle $\psi << 1$ with its axis. An atomic string in a crystal or a nanotube could be treated as such fiber.

If the particle interacts with large number of atoms within the length of radiation formation (the coherence length)

$$l_{coh} = \frac{2\gamma^2}{\omega}\frac{1}{1+\gamma^2\theta^2+(\gamma\omega_p/\omega)^2}, \quad \theta << 1, \qquad (3.1)$$

where θ is the angle between the wave vector of the radiated wave and the particle velocity, the non-uniformity of the electron density along the fiber axis is not essential for the radiation process. In this case one can use the electron density distribution in the fiber averaged along its axis:

$$n(\vec{\rho}\,') = \frac{1}{L}\int dz'\; n(\vec{r}\,'), \qquad (3.2)$$

where L is the length of the fiber, the z' axis is parallel to the fiber axis, $\vec{\rho}\,' = (x', y')$ are the coordinates in the transverse plane.

If, in addition, the conditions

$$l_{coh} >> \frac{2R}{\psi}, \quad \gamma/\omega >> R, \qquad (3.3)$$

where R is the transverse size of the fiber, are satisfied then the target can be treated as an uniform infinitely thin fiber. The electron density distribution in this case can be written using delta-function

$$n(\vec{r}\,') = n_e \delta(x')\delta(y'),$$

where n_e is the electron density per unit length of the fiber.

When the electron is incident under small angle ψ to the fiber axis, it is convenient for calculating the spectral-angular density of radiation to transform the system of coordinates (x', y', z') connected with the fiber to the system of coordinates (x, y, z) in which the z axis is parallel to the particle velocity $\vec{v}$. In the new system of coordinates the electron density distribution can be written in the form

$$n(\vec{r}\,) = n_e \delta(x - z\psi)\delta(y - y_0) \tag{3.4}$$

taking into account that the electron moves at distance y_0 to the fiber axis (see Fig.1). The coordinate y here is perpendicular to the fiber axis z' and the particle velocity vector $\vec{v}$.

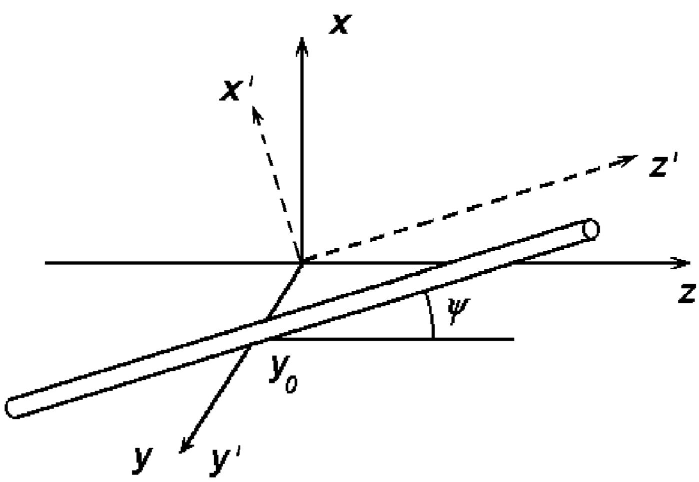

Figure 1. Target position.

Substitution (3.4) into (2.12) gives us the spectral-angular density of TR by the electron incident on the fiber with the given impact parameter y_0. It is convenient to describe the radiation by uniform flux of the particles using the radiation efficiency [6]

$$\frac{dK}{d\omega\, do} = \int dx_0 dy_0 \frac{d\mathrm{E}}{d\omega do}, \tag{3.5}$$

where x_0 and y_0 are the coordinates of the incident particles in the plane orthogonal to $\vec{v}$. In the problem under consideration when the beam is incident under small angle to the long fiber the value $d\mathrm{E}/d\omega\, do$ do not depend on x_0. In this case the radiation efficiency (3.5) could be written in the form

$$\frac{dK}{d\omega do} = L\psi \int dy_0 \frac{d\mathrm{E}}{d\omega do}. \tag{3.6}$$

Here we have used the fact that only the particles with the coordinate x_0 in the frames $\Delta x_0 = L\psi$ participate in the radiation process.

In our case the radiation efficiency could be written in the form

$$\frac{dK}{d\omega do} = \frac{Le^6 n_e^2 \gamma}{m^2 \omega \psi} F(\theta, \varphi), \tag{3.7}$$

where $F(\theta, \varphi)$ is the function that determines the angular distribution of radiation,

$$F(\theta,\varphi) = \frac{1 + 2\left(\gamma\theta\cos\varphi - \dfrac{1+\gamma^2\theta^2}{2\gamma\psi}\right)^2}{\left[1 + \left(\gamma\theta\cos\varphi - \dfrac{1+\gamma^2\theta^2}{2\gamma\psi}\right)^2\right]^{3/2}}. \tag{3.8}$$

Here φ is the azimuth angle (the angle between x axis and the projection of wave vector $\vec{k}$ to the plane (x, y)). The surface plot of the function (3.8) for $\psi = 2\cdot 10^{-3}$ rad, $\gamma = 2000$ is presented on Fig. 1 (the upper plot). It is easy to see that the angular distribution of radiation intensity possesses the axial symmetry relatively to the fiber axis ($\theta = \psi$, $\varphi = 0$), that can be demonstrated analytically from (3.8). Near the axis of symmetry the intensity of radiation is rather high. For large angles of incidence, $\psi \geq 10\gamma^{-1}$, the angular distribution of intensity takes the shape of a narrow double ring of the radius ψ. Note in connection with this, that the radiation on the fiber can be interpreted as the radiation produced by the perturbation created in the fiber by the relativistically compressed Coulomb field of the incident particle. Such perturbation moves along the fiber with the velocity exceeding the velocity of light. This situation leads to a radiation analogous to Cherenkov one.

The account of the finite value of the fiber radius R leads to the suppression of the radiation under large values of the angle θ. The middle and lower plots on Fig. 2 demonstrate the function F calculated for the case of the fiber with Gaussian distribution of the electron density in the plane orthogonal to the fiber axis,

$$n(\vec{r}) = \frac{n_e}{2\pi R^2} \exp\left[-\frac{(x - z\psi)^2 + (y - y_0)^2}{2R^2} \right], \tag{3.9}$$

for two different values of the mean square radius R.

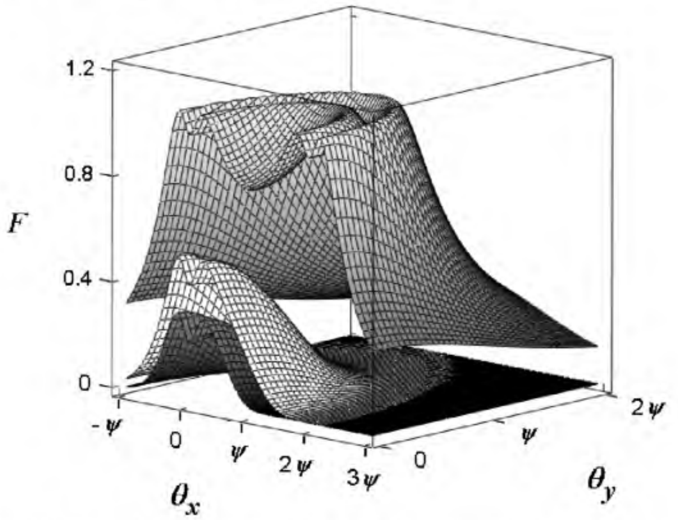

Figure 2. Surface plots of the function $F(\theta, \varphi)$ that determine the angular distribution of transition radiation of relativistic particle on thin fiber-like target for the cases of infinitely thin fiber (Eq. (3.8); upper surface) and the fiber with Gaussian distribution of the electron density (3.9) ($R\omega / \gamma = 0.1\psi\gamma$, middle surface, $R\omega / \gamma = 0.2\psi\gamma$, lower surface); $\psi = 2 \cdot 10^{-3}$ rad, $\gamma = 2000$ ($\theta_x = \theta\cos\varphi$, $\theta_y = \theta\sin\varphi$).

Note that the shape of the angular distribution of the radiation on a fiber-like target depends on the details of the electron density distribution in the fiber. The surface plots of the function F calculated for the cases of Gaussian fiber (3.9), nanotube (in the limitation case of infinitely thin walls of the tube) and uniform cylindrical fiber with the same characteristic radius R are presented on Fig. 3. For the further references, we present here the analytical expression for the function F in the case of uniform cylindrical fiber:

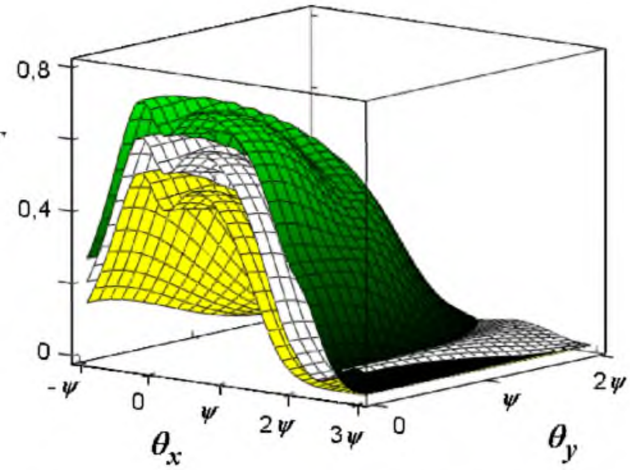

Figure 3. Surface plots of the function $F(\theta,\varphi)$ for the cases of uniform cylindrical fiber (upper surface), fiber with Gaussian distribution of the electron density (lower surface), and nanotube (middle surface) with the same characteristic radius, $R\omega/\gamma = 0.2\psi\gamma$; $\psi = 10^{-3}$ rad, $\gamma = 2000$.

$$F(\theta,\varphi) = \frac{2}{\pi}\int_{-\infty}^{\infty} dq \left(\frac{2J_1\left(\frac{R\omega}{\gamma}\sqrt{q^2 + \left(\frac{1+\gamma^2\theta^2}{2\psi\gamma}\right)^2}\right)}{\frac{R\omega}{\gamma}\sqrt{q^2 + \left(\frac{1+\gamma^2\theta^2}{2\psi\gamma}\right)^2}} \right)^2 \Phi(q,\theta,\varphi), \quad (3.10)$$

where

$$\Phi(q,\theta,\varphi) = \frac{(\gamma\theta\sin\varphi - q)^2 + \left(\gamma\theta\cos\varphi - \frac{1+\gamma^2\theta^2}{2\gamma\psi}\right)^2}{\left[1 + (\gamma\theta\sin\varphi - q)^2 + \left(\gamma\theta\cos\varphi - \frac{1+\gamma^2\theta^2}{2\gamma\psi}\right)^2\right]^2}.$$

4. EIKONAL APPROXIMATION IN TR THEORY.

As we could see in follows, Born approximation in TR theory is valid under the condition

$$\omega \frac{\omega_p^2}{\omega^2} l << 1 , \tag{4.1}$$

where l is the length on which the interaction of the particle with the non-uniformity of the dielectric permittivity happens. This inequality violates under l increase or ω decrease, and the consideration of TR process out of the range of validity of Born approximation become necessary. One of the methods that permit to work out of the frames of Born perturbation theory is the eikonal approximation.

However, direct application of eikonal approximation to TR theory leads to a number of difficulties connected with the fact that Eq. (2.1) for the electric field contains the particle's current (the last term in (2.1)) with the fact that for complex geometry of the target the problem becomes multidimensional (for instance, the problem of TR on the fiber-like target is two-dimensional problem).

The attempts to overcome these difficulties based on the construction of Green function for the equation (2.1) have been made [2,9]. The quasi classical Green function for (2.1) was built [2] in the case of one-dimensional problem on TR of relativistic particles in the medium with non-uniform dielectric permittivity. However, that method substantially uses the one-dimensional character of the problem.

The method of construction of Green function for the equation (2.1) with the particle's current that is valid under the number of conditions, which are analogous to the conditions of applicability of the eikonal approximation in quantum mechanics, was proposed [9]. Although that method permits to work out of the frames of Born perturbation theory, its application to particular problems is rather complete and awkward. Only the energy losses of the particle crossing the plate, and in random medium, were calculated on the basis of that method.

In the present article the TR theory based on the method of equivalent photons[10] is developed. In the frames of this theory the transition radiation process is considered as the process of scattering of the particle's electromagnetic field on non-uniformities of the dielectric permittivity of the medium. The field of the particle in this case is presented as the packet of free electromagnetic waves. Such approach permits to consider the equation that determine the evolution of the wave packet in non-uniform medium

$$\left(\Delta + \omega^2\right)\vec{E}_\omega = \nabla(\nabla \vec{E}_\omega) + \omega^2\left(1 - \varepsilon_\omega\right)\vec{E}_\omega \tag{4.2}$$

instead of the equation (2.1) with the particle's current. The initial state of the wave packet (before entering the medium) is the expansion of the particle's eigenfield into the set of free electromagnetic waves that permits to build the solution of Eq. (4.2) for multidimensional problem in a simple way. So, let us find the solution of Eq. (4.2) in a form

$$\vec{E}_\omega(\vec{r}) = e^{i\omega z}\vec{\Phi}(\vec{r}). \tag{4.3}$$

Let the function $\vec{\Phi}(\vec{r})$ changes itself in space slow enough to neglect its second derivations in (4.2). In this approximation the equation for $\vec{\Phi}(\vec{r})$ takes the form

$$2i\frac{\partial\vec{\Phi}}{\partial z} = -\omega\Phi_z\vec{e}_z + i\nabla\Phi_z + i\vec{e}_z\, div\,\vec{\Phi} + \omega(1-\varepsilon_\omega)\vec{\Phi}. \tag{4.4}$$

Separating longitudinal and transverse components in (4.4) we obtain

$$\frac{\partial\Phi_x}{\partial x} + \frac{\partial\Phi_y}{\partial y} = -i\omega\varepsilon_\omega\Phi_z, \tag{4.5}$$

$$2i\frac{\partial\vec{\Phi}_\perp}{\partial z} = \omega(1-\varepsilon_\omega)\vec{\Phi}_\perp + i\nabla_\perp\Phi_z. \tag{4.6}$$

Substitution Φ_z from (4.5) into (4.6) gives us

$$2i\frac{\partial\vec{\Phi}_\perp}{\partial z} = \omega(1-\varepsilon_\omega)\vec{\Phi}_\perp - \nabla_\perp\frac{1}{\omega\varepsilon_\omega}\left(\frac{\partial\Phi_x}{\partial x} + \frac{\partial\Phi_y}{\partial y}\right). \tag{4.7}$$

The second term in the right side of (4.7) could be neglected with the same precision as neglecting of the second derivations of the function $\vec{\Phi}(\vec{r})$ in (4.4) under condition

$$\rho_{eff} >> \frac{1}{\omega_p}, \tag{4.8}$$

where ρ_{eff} is the characteristic distance in transverse direction on which the value $\left|\vec{\Phi}_\perp\right|$ changes itself substantially (it is assumed also that $\varepsilon_\omega(\vec{r})$ changes itself in transverse direction slow enough to neglect the derivation $\nabla_\perp(1/\varepsilon_\omega)$). Under this condition Eq. (4.7) takes the form

$$\frac{\partial\vec{\Phi}_\perp}{\partial z} = -i\frac{\omega}{2}(1-\varepsilon_\omega)\vec{\Phi}_\perp. \tag{4.9}$$

Substituting the solution of the last equation into (4.3) we obtain

$$\vec{E}_\omega(\vec{r})_\perp = \vec{E}_\omega^{(0)}(\vec{r})_\perp \exp\left\{-i\frac{\omega}{2}\int_{-\infty}^{z}\left(1-\varepsilon_\omega(\vec{r})\right)dz\right\}, \tag{4.10}$$

where $\vec{E}_\omega^{(0)}(\vec{r})_\perp$ is the field of the incident wave packet

$$\vec{E}_\omega^{(0)}(\vec{r})_\perp = \frac{2e\omega}{v^2\gamma}e^{i\omega z}\frac{\vec{\rho}}{\rho}K_1\left(\frac{\omega\rho}{v\gamma}\right)$$

$$= \frac{ie}{\pi v}e^{i\omega z}\int\frac{\vec{\kappa}}{\omega^2-\frac{\omega^2}{v^2}-\vec{\kappa}^2}e^{i\vec{\kappa}\vec{\rho}}d^2\kappa\,. \tag{4.11}$$

It is necessary for fulfillment of the condition (4.8) that the characteristic transverse distances on which the function $\varepsilon_\omega(\vec{r})$ changes substantially would be not less than the value ρ_{eff}. In other words, our solution is valid only for the target with fuzzy boundaries.

Substituting (4.10) into (2.9) we obtain

$$\vec{I}_\perp = \frac{2e\omega}{v^2\gamma}\int d^3r\, e^{i(\omega-k_z)z}e^{-i\vec{k}_\perp\vec{\rho}}(1-\varepsilon_\omega(\vec{r}))\frac{\vec{\rho}}{\rho}K_1\left(\frac{\omega\rho}{v\gamma}\right)$$

$$\exp\left\{-i\frac{\omega}{2}\int_{-\infty}^{z}\left(1-\varepsilon_\omega(\vec{r})\right)dz\right\}. \tag{4.12}$$

The characteristic values ρ_{eff} making the main contribution into the integral (4.12) have the order

$$\rho_{eff} \sim \min\left(\gamma/\omega, 1/k_\perp\right),$$

where $k_\perp \approx \omega\theta$ and θ is the angle on which the radiation is observed ($\theta \ll 1$). So, according to (4.8), Eq. (4.12) is valid in the range of frequencies ω and radiation angles determined by the following inequalities:

$$1 \gg \frac{\omega_p}{\omega} \gg \gamma^{-1}, \qquad \frac{\omega_p}{\omega} \gg \theta\,. \tag{4.13}$$

The order of value of the argument of the first exponent in (4.12) could be estimated as $\omega\theta^2 l/2$, where l is the target thickness along the direction of the particle velocity. So under the condition

$$\frac{\omega\theta^2 l}{2} << 1 \qquad (4.14)$$

the first exponent in (4.12) could be replaced by unit. Then after integrating over z we get the following expression for $\vec{I}_\perp$:

$$\vec{I}_\perp^{(eik)} = i\frac{4e}{v^2\gamma}\int d^2\rho\, e^{-i\vec{k}_\perp\vec{\rho}}\,\frac{\vec{\rho}}{\rho}K_1\left(\frac{\omega\rho}{v\gamma}\right)$$
$$\left\{\exp\left[-i\frac{\omega}{2}\int_{-\infty}^{\infty}(1-\varepsilon_\omega(\vec{r}))dz\right]-1\right\}. \qquad (4.15)$$

Substituting it into (2.10) we obtain the spectral-angular density of the transition radiation in the eikonal approximation

$$\frac{d\mathrm{E}}{d\omega do} \approx \frac{\omega^4}{(8\pi^2)^2}\left|\vec{I}_\perp^{(eik)}\right|^2. \qquad (4.16)$$

This formula is valid in the range of frequencies and radiation angles determined by (4.13) and (4.14).

The argument of the last exponent in (4.12) could be estimated as

$$\frac{\omega}{2}\int_{-\infty}^{\infty}(1-\varepsilon_\omega(\vec{r}))\,dz \sim \omega\frac{\omega_p^2}{\omega^2}l\,.$$

If this value is small comparing to unit (that corresponds to (4.1)) then the expansion over the parameter $\omega_p^2 l/\omega$ could be made in (4.15). In the first order of that expansion the value $\vec{I}_\perp^{(eik)}$ coincides with the corresponding result of Born approximation $\vec{I}_\perp^{(B)}$.

The inequality (4.1) violates with l increase and ω decrease. Eq. (4.16) permits to describe TR also out of the range of validity of Born approximation. Indeed, the inequality $\omega_p^2 l/\omega \geq 1$ do not contradict with the conditions (4.13) and (4.14), which determine the conditions of validity of the formula (4.15). Note that (4.13) and (4.14) always could be fulfilled under the particle energy large enough in the range of characteristic transition radiation angles $\theta \sim \gamma^{-1}$.

Eq. (4.15) could be used for consideration of TR on the target of complex configuration, such as dielectric fiber.

5. TR IN A THIN LAYER OF SUBSTANCE

Consider TR under normal incidence of the ultrarelativistic particle onto thin uniform plate with the thickness a as the simplest example of application of eikonal approximation. In this simplest one-dimensional case the dielectric properties of the target do not depend on ρ, so the condition (4.8) is fulfilled automatically. Computation using (4.15) leads to

$$\vec{I}_{\perp}^{(eik)} = \frac{8\pi e}{v\omega} \frac{\vec{k}_{\perp}}{k_{\perp}^2 + \left(\frac{\omega}{v\gamma}\right)^2} \left\{ \exp\left[-i\frac{\omega}{2}\frac{\omega_p^2}{\omega^2} a \right] - 1 \right\}, \tag{5.1}$$

and for the spectral-angular density of the radiation on small angles we obtain

$$\frac{dE}{d\omega do} = 2\frac{e^2}{\pi^2} \frac{\theta^2}{\left(\theta^2 + \gamma^{-2}\right)^2} \left\{ 1 - \cos\left[\frac{\omega_p^2}{\omega^2}\frac{\omega a}{2} \right] \right\}. \tag{5.2}$$

Compare the results obtained in Born (2.13), (2.14) and eikonal (5.1), (5.2) approximations with the precise formula for the spectral-angular density of TR on the thin plate. The last one in the range of small radiation angles has the form[1]

$$\frac{dE}{d\omega do} = 2\frac{e^2}{\pi^2} \left(\frac{\omega_p^2}{\omega^2}\right)^2 \frac{\theta^2}{\left(\theta^2 + \gamma^{-2}\right)^2 \left(\theta^2 + \gamma^{-2} + \frac{\omega_p^2}{\omega^2}\right)^2}$$

$$\times \left\{ 1 - \cos\left[\left(\theta^2 + \gamma^{-2} + \frac{\omega_p^2}{\omega^2} \right) \frac{\omega a}{2} \right] \right\}. \tag{5.3}$$

One could obtain this result by substitution into (2.9) the precise expression for the electric field inside the plate, which could be found via “sewing” of the solutions of Eqs. (2.1), (2.2) on the boundaries of the plate. It happens that one could satisfy the boundary conditions only adding to the solutions of Eqs. (2.1), (2.2) with the particle’s charge and current the solutions of free equations that correspond to the radiation field. The procedure described leads to the following formula for the total electric field in the medium:

$$\vec{E}_\omega(\vec{r})_\perp = \frac{ie}{\pi v}\int d^2\kappa\, e^{i\vec{\kappa}\vec{\rho}}\vec{\kappa}\times \tag{5.4}$$

$$\times\left\{\frac{e^{i\frac{\omega}{v}z}}{\varepsilon_\omega\left(\omega^2\varepsilon_\omega-\frac{\omega^2}{v^2}-\vec{\kappa}^2\right)}+\frac{e^{iz\sqrt{\varepsilon_\omega\omega^2-\vec{\kappa}^2}}}{\omega^2-\frac{\omega^2}{v^2}-\vec{\kappa}^2}-\frac{e^{iz\sqrt{\varepsilon_\omega\omega^2-\vec{\kappa}^2}}}{\varepsilon_\omega\left(\omega^2\varepsilon_\omega-\frac{\omega^2}{v^2}-\vec{\kappa}^2\right)}\right\}.$$

Substituting this formula into (2.9) we obtain

$$\vec{I}_\perp = \frac{4\pi ie}{v}(1-\varepsilon_\omega)\vec{k}_\perp\int_0^L dz\, e^{-ik_z z}\times \tag{5.5}$$

$$\times\left\{\frac{e^{i\frac{\omega}{v}z}}{\varepsilon_\omega\left(\omega^2\varepsilon_\omega-\frac{\omega^2}{v^2}-\vec{k}_\perp^2\right)}+\frac{e^{iz\sqrt{\varepsilon_\omega\omega^2-\vec{k}_\perp^2}}}{\omega^2-\frac{\omega^2}{v^2}-\vec{k}_\perp^2}-\frac{e^{iz\sqrt{\varepsilon_\omega\omega^2-\vec{k}_\perp^2}}}{\varepsilon_\omega\left(\omega^2\varepsilon_\omega-\frac{\omega^2}{v^2}-\vec{k}_\perp^2\right)}\right\}.$$

For the dielectric permittivity in the form (2.11) and for small radiation angles Eq. (5.5) takes the form

$$\vec{I}_\perp = \frac{4\pi ie}{v}\frac{\omega_p^2}{\omega^2}\vec{k}_\perp\int_0^L dz\, e^{-ik_z z}\times \tag{5.6}$$

$$\left\{-\frac{e^{i\frac{\omega}{v}z}}{\left(\omega^2\gamma^{-2}+\omega_p^2+\omega^2\theta^2\right)}-\frac{e^{iz\omega\left(1-\frac{\omega_p^2}{2\omega^2}-\frac{\theta^2}{2}\right)}}{\omega^2\gamma^{-2}+\omega^2\theta^2}+\frac{e^{iz\omega\left(1-\frac{\omega_p^2}{2\omega^2}-\frac{\theta^2}{2}\right)}}{\left(\omega^2\gamma^{-2}+\omega_p^2+\omega^2\theta^2\right)}\right\}.$$

Substituting the last result into (2.10) we obtain (5.3).

Under conditions

$$\gamma^2\frac{\omega_p^2}{\omega^2}<<1, \tag{5.7}$$

$$\omega\frac{\omega_p^2}{\omega^2}a<<1 \tag{5.8}$$

the precise result for the TR intensity (5.3) transforms into (2.14) that corresponds to Born approximation.

Under conditions

$$\left(\theta^2, \gamma^{-2}\right) << \frac{\omega_p^2}{\omega^2} << 1 \qquad (5.9)$$

the precise result (5.3) transforms into the formula (5.2) corresponding to eikonal approximation.

6. TR ON THE FIBER-LIKE TARGET IN EIKONAL APPROXIMATION

Consider now TR arising under incidence of fast charged particles on dielectric fiber-like target under small angle ψ to the fiber axis. Let the fiber has the cylindrical shape with radius R and uniform distribution of the electron density, as an example. The effective target thickness along the particle's motion direction in this case is $l \sim 2R/\psi$, so the condition (4.14) takes the form

$$\frac{\omega \theta^2 R}{\psi} << 1 \, . \qquad (6.1)$$

The formula (4.15) in this case gives us

$$\vec{I}_\perp^{(eik)} = \frac{4\pi i e}{v\omega} \int_{-\infty}^{\infty} dy\, e^{-ik_y y} e^{-|y|\sqrt{k_x^2 + (\omega/v\gamma)^2}} \left(\frac{-ik_x \vec{e}_x}{\sqrt{k_x^2 + (\omega/v\gamma)^2}} + \vec{e}_y \operatorname{sgn} y \right) \times$$

$$\times \left\{ \exp\left[-i \frac{\omega}{2} \frac{\omega_p^2}{\omega^2} \frac{2}{\psi} \sqrt{R^2 - (y - y_0)^2} \right] - 1 \right\}, \qquad (6.2)$$

where the z axis is parallel to the particle velocity $\vec{v}$, the fiber axis is parallel to the plane (x, z), y_0 is the impact parameter of the incident particle in relation to the fiber axis. Consider some limitation cases of the formula obtained.

At first, let us find the conditions under which the results of Born and eikonal approximations coincide to each other. As it was mentioned at the end of part 3, the necessary condition of such coincidence is the smallness of the effective target thickness along the particle's motion direction $l \sim 2R/\psi$, that gives the possibility to make an expansion of the last exponent in (6.2) over the parameter

$$\omega \frac{\omega_p^2}{\omega^2} \frac{2R}{\psi} << 1 . \qquad (6.3)$$

In other words, it is necessary to outspread the condition of validity of Born approximation (5.8) to the result obtained in the eikonal approximation.

On the other hand, it is necessary to outspread the condition of smallness of radiation angles (6.1) to the corresponding Born result

$$\vec{I}_\perp^{(B)} = \frac{4\pi e}{v} \frac{\omega_p^2}{\omega^2} \int_{y_0-R}^{y_0+R} dy\, e^{-ik_y y} e^{-|y|\sqrt{\frac{\omega^2}{\gamma^2}+\left(k_x - \frac{1}{\psi}\left(\frac{\omega}{v}-k_z\right)\right)^2}} \times$$

$$\times \left(-i\vec{e}_x \frac{k_x - \frac{1}{\psi}\left(\frac{\omega}{v} - k_z\right)}{\sqrt{\frac{\omega^2}{\gamma^2} + \left(k_x - \frac{1}{\psi}\left(\frac{\omega}{v} - k_z\right)\right)^2}} + \vec{e}_y \operatorname{sgn} y \right) \times \qquad (6.4)$$

$$\times \frac{\sin\left(\frac{1}{\psi}\left(\frac{\omega}{v} - k_z\right)\sqrt{R^2 - (y - y_0)^2}\right)}{\frac{\omega}{v} - k_z} .$$

It is easy to see that in the first order of the expansion over the small parameters (6.1) and (6.3), under additional condition

$$\psi \gamma >> 1 , \qquad (6.5)$$

the expressions for the value $\vec{I}_\perp$ that describe the properties of TR on the fiber in Born and eikonal approximation will coincide:

$$\vec{I}_\perp^{(eik)} \approx \vec{I}_\perp^{(B)} \approx \frac{4\pi e}{v} \frac{\omega_p^2}{\omega^2} \int_{y_0-R}^{y_0+R} dy\, e^{-ik_y y} e^{-|y|\sqrt{k_x^2 + (\omega/\gamma)^2}} \times \qquad (6.6)$$

$$\times\left(\frac{-ik_x\vec{e}_x}{\sqrt{k_x^2+(\omega/\gamma)^2}}+\vec{e}_y\,\mathrm{sgn}\,y\right)\frac{1}{\psi}\sqrt{R^2-(y-y_0)^2}\,.$$

So, when the conditions (6.1), (6.3) and (6.5) are satisfied, the formulae that describe TR on the fiber in Born approximation are justified not only in the range of frequencies determined by (5.7), but also in more soft range of the radiation spectrum, where the eikonal approximation is applicable (see the condition (5.9)).

Substitution of the value $\vec{I}_\perp$ into (2.10) gives us the spectral-angular density of TR by the electron incident on the fiber with the given impact parameter y_0. After integration over impact parameters we obtain the radiation efficiency in the form (3.7). The surface plot of the function F in the case when the conditions (6.1), (6.3) and (6.5) are satisfied is presented on Fig. 4. The distinct between the results of calculations using the "precise" formula (3.10) and approximated (6.6) is not more than 10 %.

Consider now the limitation case of thick fiber, when its radius R is large comparing to the characteristic transverse size of the Coulomb field of the incident particle $\sim\gamma/\omega$, where the Fourier components of the Coulomb field with the frequency ω are concentrated:

$$R>>\gamma/\omega\,. \tag{6.7}$$

Note that the characteristic values of y making the main contribution into the integral (6.2) are of the order of value γ/ω whereas the characteristic values of y_0 have the order of value R. In the limitation case of thick fiber (6.7) one could neglect the dependence of the last exponent in (6.2) on y. In this case

$$\vec{I}_\perp^{(eik)}\approx i\frac{4\pi e}{v\omega}\left\{\exp\left[-i\frac{\omega}{2}\frac{\omega_p^2}{\omega^2}\frac{2}{\psi}\sqrt{R^2-y_0{}^2}\right]-1\right\}\times$$

$$\times\int_{-\infty}^{\infty}dy\,e^{-ik_y y}e^{-|y|\sqrt{k_x^2+(\omega/\gamma)^2}}\left(\frac{-ik_x\vec{e}_x}{\sqrt{k_x^2+(\omega/\gamma)^2}}+\vec{e}_y\,\mathrm{sgn}\,y\right)= \tag{6.8}$$

$$=\frac{8\pi e}{v\omega}\frac{\vec{k}_\perp}{k_\perp^2+(\omega/\gamma)^2}\left\{\exp\left[-i\frac{\omega}{2}\frac{\omega_p^2}{\omega^2}\frac{2}{\psi}\sqrt{R^2-y_0{}^2}\right]-1\right\}.$$

This result coincides to the formula (5.1) for TR under normal incidence of the particle onto the plate of the thickness $a = 2\sqrt{R^2 - y_0^2} / \psi$ on which the particle effectively interacts with the fiber under the given value of the impact parameter y_0. The analogous result could be obtained also for fibers with inhomogeneous distribution of the electron density in the plane perpendicular to the fiber's axis.

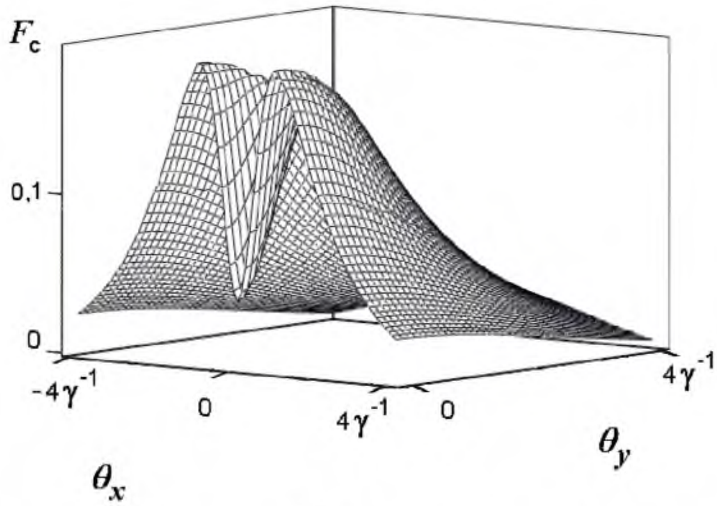

Figure 4. Surface plot of the function $F(\theta,\varphi)$ for the uniform cylindrical fiber in the case when the conditions (6.1), (6.3), (6.5) are satisfied: $\gamma = 2000$, $\psi = 0.1$ rad, $R\omega/\gamma = 0.015\psi\gamma$, $\gamma\omega_p/\omega \leq 3$ (for example: $R = 10^{-5}$ cm, $\hbar\omega_p = 20$ eV, $\hbar\omega = 13$ keV).

So, in the limitation case $R >> \gamma/\omega$ TR on the fiber is equivalent to that on the uniform plate with the thickness and electron density determined by the local effective thickness and average density of the fiber under the given value of y_0.

7. CONCLUSION

The process of transition radiation of relativistic particles in the medium with non-uniform dielectric permittivity is considered. The approach to description of this process based on the equivalent photons method and eikonal approximation is proposed. The general formulae for TR spectral-angular density for the arbitrary form of the dielectric function of the medium are obtained. That formulae permit to consider TR in the range of small radiation angles in the case when the dielectric permittivity depends on more than one coordinate. It is demonstrated that application of eikonal approximation for that problem makes possible the work out of the frames of Born approximation, which uses the expansion of the radiation fields by the small deviation of the dielectric permittivity from unit. The conditions of validity of Born and eikonal approximations in the problem under consideration are obtained.

TR under the incidence of the particles on the fiber-like target was considered as an example of usage of our formulae in multidimensional problems. Spectral-angular densities calculated for this case using Born and eikonal approximations are compared. The conditions under which the work out of the frames of Born approximation is necessary are obtained.

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