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A Mathematical Model of Interaction of the Results of Different R&D Types

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Abstract—A three-dimensional model of exchange among different R&D types, i.e., basic articles, applied articles, and patents on inventions, has been constructed and investigated within the linear concept of innovations and equations of population dynamics. With linear functions for growth coefficients from phase variables, all eight critical points of the constructed third-order dynamic system were found.
The preconditions for the stability of the nontrivial critical point have been determined. Similar results are obtained for nonlinear self-contained functions of the growth coefficient without a decrease of knowledge.

Keywords: linear concept of innovations, equations of population dynamics, basic papers, applied papers, patents on inventions, dynamic system, linear analysis of stability, critical points, Jacobian matrix.

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In this article, a three-dimensional model of the interaction of results of different R&D types: basic papers, applied papers, and patents for inventions, is constructed and analyzed within a linear model of innovations and equations of population dynamics.

The analogy between the considered linear innovation process and the dynamics of populations is as follows. In population biology (ecology), a population is understood as the totality of a certain species. Populations of different types that consist of species of different varieties can interact with each other in different ways. Interactions of competition, cooperation, "predator–prey," mutualism, parasitism, and other types occur. But all of these, during the interaction of n-populations and the constancy of model coefficients, are described by the Lotka–Volterra [1] equations.

$$\frac{dx_i}{dt} = x_i\left[\alpha_i - \sum_{j=1}^{n}\gamma_{ij}x_j\right]. \qquad (1)$$

The type of interaction is determined by a combination of signs or zero values of constant coefficients of a model. For example, if $\alpha_i > 0$, $\gamma_{ij} > 0$, then we come to the classical n-dimensional model of competitive interactions of populations. By analogy with the concept of a biological population, the "populations" of the innovation process are understood here as the number of basic (x_1) and applied (x_2) papers, and patents (x_3).

As is seen from the n-dimensional dynamic system (1), the self-development at each stage of the innovation process is described by standard logistic terms ($\alpha_i x_i - \gamma_{ii}x_i^2$) or by the Verhulst equation $\frac{dx_i}{dt} = \alpha_i x_i - \gamma_{ii}x_i^2$. Here, in the absence of paired interactions $x_i x_j$ ($i \neq j$), the self-development process at each ith scientific-research stage comes to a stable stationary level $\xi_{ist} = \frac{\alpha_i}{\gamma_{ii}}$. In scientometrics (the dynamics of publications and patents on inventions) and in the models of innovative diffusion, the use of the Verhulst equation is generally accepted [2, 3]. The paired interactions (bilinear terms) in equations of population dynamics (1) are interpreted by the probability of encounters among the species of different varieties (the formula of the product of probabilities). In a similar way, in our case, for example, the term $\gamma_{21}x_1x_2$ in dynamic system (1) can be interpreted as the probability of encounters of complementary (mutually appropriate) basic and applied papers leading to new applied papers (basic research enriches applied work with new knowledge, which leads to new applied research).

The equations of population dynamics, like their analogues in innovation and economic dynamics [2, 4], are a system of balance equations, in which the rates of total increase of populations are composed of the rates of their true increase (plus sign) and decrease (minus sign). Despite the "physicality" of these equations (balance equations), their right-hand parts are of a phenomenological character because they are not inferred from basic principles and laws (this is especially true of economic dynamics). The general recognition of these equations in biology and ecology is based on their experimental verification carried out by the Russian biologist G. Gause [5], who experimentally validated the mathematical model of competitive interactions (the Lotka–Volterra equation). It should be noted that a number of innovation diffusion models relating to the Verhulst equation also have been validated on the basis of empirical evidence (the works of American researchers Mansfield, Sagal, and others [2]).

Now, we can construct a three-dimensional model of interaction of the results of R&D of different varieties. In terms of the population dynamics equations and of the linear concept of innovations, such a model is suggested as:

$$\begin{cases} \frac{dx}{dt} = f_1(y)x - k_1x - \beta_1x^2; \\ \frac{dy}{dt} = f_2(x)y - k_2y - \beta_2y^2; \\ \frac{dz}{dt} = f_3(y)z - k_3z - \beta_3z^2, \end{cases} \quad (2)$$

where x is the number of published basic articles, y is the number of published applied articles, z is the number of issued patents on inventions, k_i is the coefficient of obsolescence of R&D of different types, β_i is the coefficient of intraspecific competition among the results of R&D of different kinds, f_i are variable growth coefficients in terms that are responsible for the generation of results from R&D of different kinds.

The considered scientific-research system is assumed to contain R&D works of a natural scientific and technological character whose results are well described by the consecutive series: basic papers, applied papers, and patents on inventions.

The first growth coefficient $f_1(y)$ shows that the rate of generation of basic articles depends on the number of applied articles. In doing basic research and in writing the related articles, researchers make use of the existing applied knowledge and make references to applied articles in their articles. We take the example of a researcher who comes across an applied article (or a series of such articles) in the array of earlier basic papers (written by himself and by other scientists), prepares his basic research on the basis of all these works and writes a series of new basic articles. Here applied knowledge enriches basic knowledge. The same thing takes place in the second equation of the dynamic system, where, in contrast, basic knowledge enriches applied knowledge. All this is seen in references to applied works in basic papers and in references to basic works in applied papers.

We also assume that the existence of patents on inventions is unaffected by the increase of applied papers. In fact, references to patents are rare in applied papers. If in certain research domains this is not so, the function $f_2(x, z)$ should be introduced.

In writing the third equation we assume that the wording in descriptions of inventions is affected exclusively by applied knowledge. In descriptions of inventions we actually do not see references to basic papers.

In the simplest case, as in the equations of population dynamics, it is natural to assign a linear form to the function f_i, then we come to the dynamic system.

$$\begin{cases} \frac{dx}{dt} = (\alpha_1 + b_1y)x - k_1x - \beta_1x^2; \\ \frac{dy}{dt} = (\alpha_2 + b_2x)y - k_2y - \beta_2y^2; \\ \frac{dz}{dt} = (\alpha_3 + b_3y)z - k_3z - \beta_3z^2. \end{cases} \quad (3)$$

The terms that are related to the generation of results of different R&D types fit well into the concept of mutual enrichment and complementarity of knowledge in the preparation of R&D results for publication. For example, the term responsible for the generation of basic articles is divided into two parts: $\alpha_1x + b_1yx$ where the former is responsible for self-generation of basic works (basic articles reproduce themselves according to the Malthusian law), and the latter for mutual enrichment of basic and applied knowledge (the probability of several complementary basic and applied publications leading to new basic or applied works is an analogue with the occurrence of predators and prey in the Lotka–Volterra model).

If, as has been noted, it is necessary to introduce the growth coefficient of applied papers in dependence on both the number of basic papers and the number of descriptions of patents on inventions, then in linear approximation it will be: $f_2(x, z) = a_2 + b_2x + c_2z$.

In dynamic system (3), it is possible, by assembling linear terms, to identify $\overline{a_i} = a_i - k_i$ coefficients similar to the coefficients of total growth in demography (the difference between fertility and mortality rates). Now, it is possible to start the mathematical analysis of dynamic system (3).

All of its eight critical points are obtained as:

1. $x^* = 0, \quad y^* = 0, \quad z^* = 0;$

2. $x^* = 0, \quad y^* = 0, \quad z^* = \frac{a_3 - k_3}{\beta_3};$

3. $x^* = 0, \quad y^* = \frac{a_2 - k_2}{\beta_2}, \quad z^* = 0;$

4. $x^* = \frac{a_1 - k_1}{\beta_1}, \quad y^* = 0, \quad z^* = 0;$

5. $x^* = 0, y^* = \frac{a_2 - k_2}{\beta_2},$

$$z^* = \frac{\beta_2(a_3 - k_3) + b_3(a_2 - k_2)}{\beta_2\beta_3};$$

6. $x^* = \dfrac{a_1 - k_1}{\beta_1}, \quad y^* = 0, \quad z^* = \dfrac{a_3 - k_3}{\beta_3}$;

7. $x^* = \dfrac{\beta_2(a_1 - k_1) + b_1(a_2 - k_2)}{\beta_1\beta_2 - b_1 b_2}$,

$y^* = \dfrac{\beta_1(a_2 - k_2) + b_2(a_1 - k_1)}{\beta_1\beta_2 - b_1 b_2}, \quad z^* = 0$;

8. $x^* = \dfrac{\beta_2(a_1 - k_1) + b_1(a_2 - k_2)}{\beta_1\beta_2 - b_1 b_2}$,

$y^* = \dfrac{\beta_1(a_2 - k_2) + b_2(a_1 - k_1)}{\beta_1\beta_2 - b_1 b_2}$, $z^* =$
$\dfrac{b_2 b_3(a_1 - k_1) + \beta_1 b_3(a_2 - k_2) + (a_3 - k_3)(\beta_1\beta_2 - b_1 b_2)}{\beta_3(\beta_1\beta_2 - b_1 b_2)}$;

The Jacobian matrix of the linearized dynamic system (3) in the arbitrary critical point is:

$$A = \begin{pmatrix} a_1 - k_1 + b_1 y^* - 2\beta_1 x^* & b_1 x^* & 0 \\ b_2 y^* & a_2 - k_2 + b_2 x^* - 2\beta_2 y^* & 0 \\ 0 & b_3 z^* & a_3 - k_3 + b_3 y^* - 2\beta_3 z^* \end{pmatrix}. \quad (4)$$

For the eighth nontrivial critical point this is simplified to:

$$A = \begin{pmatrix} -\beta_1 x^* & b_1 x^* & 0 \\ b_2 y^* & -\beta_2 y^* & 0 \\ 0 & b_3 z^* & -\beta_3 z^* \end{pmatrix}. \quad (5)$$

The characteristic equation at this point after all transformations are written is:

$$|A - \lambda I| = (\beta_3 z^* + \lambda) \times [b_1 b_2 x^* y^* - (\beta_1 x^* + \lambda)(\beta_2 y^* + \lambda)] = 0. \quad (6)$$

From this we will obtain the eigenvalues of the Jacobian matrix:

$$\lambda_{1,2} = -\frac{(\beta_1 x^* + \beta_2 y^*)}{2} \pm \sqrt{\frac{(\beta_1 x^* + \beta_2 y^*)^2}{4} - (\beta_1\beta_2 - b_1 b_2)x^* y^*}$$
$$= \frac{-(\beta_1 x^* + \beta_2 y^*) \pm \sqrt{(\beta_1 x^* - \beta_2 y^*)^2 + 4 b_1 b_2 x^* y^*}}{2},$$
$$\lambda_3 = -\beta_3 z^*.$$

The analysis of the expressions of two first eigenvalues with account for $\lambda_3 = -\beta_3 z^* < 0$ leads to the following conditions:

(1) with $\beta_1\beta_2 - b_1 b_2 > 0$ we have $\lambda_{1,2} < 0$;

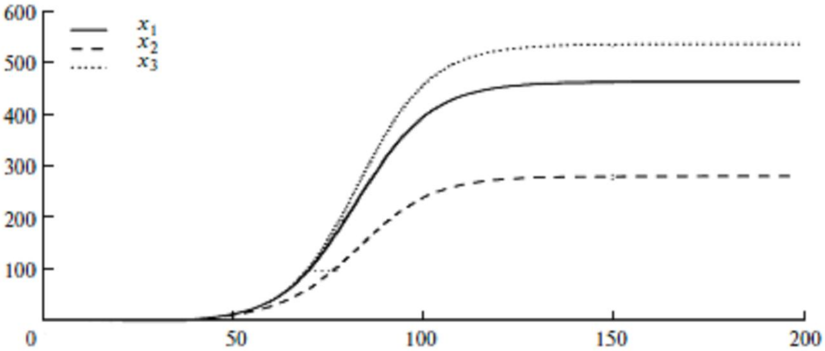
(2) with $\beta_1\beta_2 - b_1 b_2 < 0$we have $\lambda_1 < 0, \lambda_2 > 0$.

In the former case this is a stable node, in the latter this is a saddle. Considering that in a normal process the generation of the results of various R&D types is $a_i - k_i > 0$, from the expressions for the nontrivial critical point $\beta_1\beta_2 - b_1 b_2 > 0$ (under the condition of the positiveness of its coordinates) follows. This fits well into the former case of the stability of the nontrivial critical point.

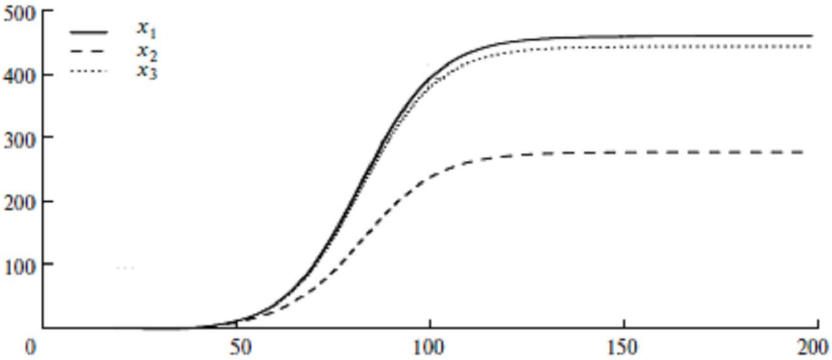
This analysis is of interest because it does not require the substitution into λ of explicit and cumbersome expressions for the coordinates of the nontrivial critical point. For the remaining seven critical points we write at once the eigenvalues of the Jacobian matrices (the numeration and expressions for the coordinates are given above):

1. $\lambda_1 = a_1 - k_1, \quad \lambda_2 = a_2 - k_2, \quad \lambda_3 = a_3 - k_3$;
2. $\lambda_1 = a_1 - k_1, \quad \lambda_2 = a_2 - k_2, \quad \lambda_3 = k_3 - a_3$;
3. $\lambda_1 = \dfrac{\beta_2(a_1 - k_1) + b_1(a_2 - k_2)}{\beta_2}, \quad \lambda_2 = k_2 - a_2$,
$\lambda_3 = \dfrac{\beta_2(a_3 - k_3) + b_3(a_2 - k_2)}{\beta_2}$;
4. $\lambda_1 = k_1 - a_1, \quad \lambda_2 = \dfrac{\beta_2(a_1 - k_1) + b_1(a_2 - k_2)}{\beta_2}$,
$\lambda_3 = a_3 - k_3$;
5. $\lambda_1 = \dfrac{\beta_2(a_1 - k_1) + b_1(a_2 - k_2)}{\beta_2}, \quad \lambda_2 = k_2 - a_2$,
$\lambda_3 = \dfrac{-\beta_2(a_3 - k_3) - b_3(a_2 - k_2)}{\beta_2}$;
6. $\lambda_1 = k_1 - a_1, \quad \lambda_2 = \dfrac{\beta_1(a_2 - k_2) + b_2(a_1 - k_1)}{\beta_1}$,
$\lambda_3 = k_3 - a_3$;
7. $\lambda_3 =$
$\dfrac{(a_3 - k_3)(\beta_1\beta_2 - b_1 b_2) + b_3\beta_1(a_2 - k_2) + b_3 b_2(a_1 - b_1)}{\beta_1\beta_2 - b_1 b_2}$,
$\lambda_{1,2}$ are analogous to expressions for the eighth critical point.

In the normal course of the scientific research process where the total coefficient of the growth of publi-



Model parameters and calculated coordinates of the nontrivial critical point:
$\alpha_1 - 0.2, \alpha_2 - 0.2, \alpha_3 - 0.2, b_1 - 0.008, b_2 - 0.001, b_3 - 0.0006,$
$k_1 - 0.1, k_2 - 0.1, k_3 - 0.1, \beta_1 - 0.0007, \beta_2 - 0.002, \beta_3 - 0.0005,$
$x_1 - 467, x_2 - 283, x_3 - 540$



Model parameters and calculated coordinates of the nontrivial critical point:
$\alpha_1 - 0.2, \alpha_2 - 0.2, \alpha_3 - 0.2, b_1 - 0.008, b_2 - 0.001, b_3 - 0.0006,$
$k_1 - 0.1, k_2 - 0.1, k_3 - 0.1, \beta_1 - 0.0007, \beta_2 - 0.002, \beta_3 - 0.0006,$
$x_1 - 467, x_2 - 283, x_3 - 450$

Results of the numerical experiment by model (3).

cations and patents on inventions is positive ($a_i - k_i > 0$) we come to the following character of the stability of critical points: point 1 is an unstable node, points 2–7 are saddles.

Thus, all seven critical points that have from one to three nulls are unstable and the only stable point, under certain conditions, can be the eighth nontrivial point.

We consider the case of an exhausted domain of research that is at the stage of dying (k_i is large), or the case of a deadlocked study ($a_i \approx 0$): $a_i - k_i < 0$, then point 1 is a stable node and points 2–7 are saddles. Therefore, we obtain a reasonable result:

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t) = 0.$$

From the expressions for the coordinates of the nontrivial critical point that depend on the relationships of the model parameters it is possible to obtain various relationships for its coordinates. For example, from $\beta_2 > b_2$, $b_1 > \beta_1$ it follows that $x^* > y^*$ (in the stationary case the number of basic papers exceeds the number of applied papers).

Taking into account that $z^* = \frac{b_3}{\beta_3} y^* + \left(\frac{a_3 - k_3}{\beta_3}\right)$, from inequality $b_3 > \beta_3$ we obtain at once $z^* > y^*$. It is clear that beginning with a certain critical value b_3 dependent on $(a_3 - k_3)/\beta_3$ and fulfilling the inequality $b_3 < \beta_3$, the opposite inequality $z^* < y^*$ will be fulfilled (the number of the applied papers exceeds the number of patents on inventions).

Here we give the results of several numerical experiments with model (3) illustrating the foregoing relationships between the coordinates of the nontrivial critical point (see the figure).

If $k_i = 0$ and $\beta_i \neq 0$, then we have the same conditions for the stability of the nontrivial critical point.

If $k_i \neq 0$, $\beta_i = 0$, then with $a_i - k_i > 0$ the zero critical point is an unstable node (infinitely growing time-dependent solutions), with $a_i - k_i < 0$ the zero critical point is a stable node (tending to zero time-dependent solutions), the critical point $x^* = (k_2 - a_2)/b_2$, $y^* = (k_1 - a_1)/b_1$, for the system of the first two equations (3) is a saddle, that is, an unstable one.

If $k_i = 0$, $\beta_i = 0$, then with $a_i \neq 0$ the zero critical point is an unstable node (in the absence of direct and indirect losses of relevance and significance of the R&D results there are no barriers to their unlimited increase) and with $a_i = 0$ the analytical solution of the system of the first two equations (3) with $t \longrightarrow \infty$ leads to the following limit solutions:

(1) with $b_1y_0 - b_2x_0 > 0$ have $\lim\limits_{t\to\infty} x(t) = x_0 - b_1y_0/b_2 < 0$, $\lim\limits_{t\to\infty} y(t) = 0$;

(2) with $b_1y_0 - b_2x_0 < 0$ have $\lim\limits_{t\to\infty} x(t) = 0$, $\lim\limits_{t\to\infty} y(t) = y_0 - (b_2/b_1)x_0 < 0$, where $x(t = 0) = x_0$, $y(t = 0) = y_0$ starting conditions of equation system (3).

These last result show that in the absence of self-reproduction of the results of basic ($a_1 = 0$) and applied ($a_2 = 0$) studies such research cannot develop.

The suggested model can be calibrated and tested using the data of empirical scientometric and patentometric studies, although the task of division of the entire flow of research studies (articles) into basic and applied is rather complicated.

The variable growth coefficients in the equations of population dynamics, as well as chemical and biophysical kinetics, are often assigned a function describing the self-limited increase of these coefficients rather than a linear function. Usually this coefficient is assigned by the function $f(x) = \frac{bx}{c + dx}$, where b, c, d are positive constants, then

$$\lim_{x\to\infty} f(x) = \frac{b}{d}.$$

Assuming this standard form of the variables of the growth coefficient also in our starting dynamic system, we come to the following model

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{b_1yx}{c_1 + d_1y} - k_1x - \beta_1x^2; \\ \dfrac{dy}{dt} = \dfrac{b_2xy}{c_2 + d_2y} - k_2y - \beta_2y^2; \\ \dfrac{dz}{dt} = \dfrac{b_3yz}{c_3 + d_3y} - k_3z - \beta_3z^2. \end{cases} \quad (7)$$

Unlike the previous dynamic system with linear coefficients f_i, the terms that are responsible for the self-reproduction of articles and patents on inventions are naturally absent here.

In the case of the influence on the growth coefficient of applied articles not only by basic articles but by patents on inventions, it is possible to introduce the function

$$f_2(x, z) = \frac{b_2xz}{c_2 + d_2xz}.$$

To calculate the critical points of dynamic system (7) by the analytical method is difficult. In the assumption of $k_i = 0$, the coordinates of the nontrivial critical point are obtained as

$$\begin{cases} x^* = \dfrac{b_1b_2 - c_1c_2\beta_1\beta_2}{\beta_1(b_2d_1 + c_1d_2\beta_2)},\ y^* = \dfrac{b_1b_2 - c_1c_2\beta_1\beta_2}{\beta_2(b_1d_2 + c_2d_1\beta_1)}; \\ z^* = \dfrac{b_3(b_1b_2 - c_1c_2\beta_1\beta_2)}{c_3\beta_2\beta_3(b_1d_2 + c_2d_1\beta_1) + d_3\beta_3(b_1b_2 - c_1c_2\beta_1\beta_2)}. \end{cases} \quad (8)$$

For this point we have obtained a characteristic equation of the Jacobian matrix of linearized system (7) with account for $k_i \neq 0$

$$(-\beta z^* - \lambda)\left[\lambda^2 + (\beta_1x^* + \beta_2y^*) + \beta_1\beta_2x^*y^* - \frac{b_1b_2c_1c_2x^*y^*}{(c_2 + d_2x^*)^2(c_1 + d_1y^*)^2}\right] = 0, \quad (9)$$

from which follows $\lambda_1 = -\beta_3 z^* < 0$,

$$\lambda_{2,3} = -\frac{(\beta_1 x^* + \beta_2 y^*)}{2} \pm \sqrt{\frac{(\beta_1 x^* + \beta_2 y^*)^2}{4} - \left[\beta_1\beta_2 - \frac{b_1 b_2 c_1 c_2}{(c_2 + d_2 x^*)^2 (c_1 + d_1 y^*)^2}\right] x^* y^*}$$

$$= -\frac{(\beta_1 x^* + \beta_2 y^*)}{2} \pm \sqrt{\frac{(\beta_1 x^* - \beta_2 y^*)^2}{4} + \frac{b_1 b_2 c_1 c_2 x^* y^*}{(c_2 + d_2 x^*)^2 (c_1 + d_1 y^*)^2}}.$$

Of the two expressions for $\lambda_{2,3}$ with account for $\lambda_1 < 0$ with $\beta_1\beta_2 - \frac{b_1 b_2 c_1 c_2}{(c_2 + d_2 x^*)^2 (c_1 + d_1 y^*)^2} > 0$ we have a stable node ($\lambda_2 < 0$, $\lambda_3 < 0$) and with the opposite inequality, the saddle ($\lambda_2 < 0$, $\lambda_3 > 0$).

We note again that these calculations are made with account for $k_i \neq 0$ but the coordinates of the critical nontrivial point are obtained with $k_i = 0$ (8).

In addition to the nontrivial critical point, the dynamic system (7) with $k_i = 0$ has a zero critical point ($x^* = y^* = z^* = 0$) and the critical point $x^* \neq 0$, $y^* \neq 0$, $z^* = 0$, where x^*, y^* is found from expressions (8). The zero critical point leads to the degenerate characteristic equation: $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and the second critical point with a single zero coordinate leads to the characteristic equation (9), where instead of the multiplier in round brackets $(-\beta_3 z^* - \lambda)$, occurs $\left(\frac{\beta_3 y^*}{c_3 + d_3 y^*} - \lambda\right)$. From this characteristic equation it follows that $\lambda_1 > 0$, and, therefore, the considered critical point is unstable.

Thus, in both considered cases, with linear and nonlinear functions f_i, in the nontrivial critical point the conditions for the appearance of a stable node and an unstable saddle were obtained.

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