Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 310, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

RIEMANN-HILBERT PROBLEM FOR THE MOISIL-TEODORESCU SYSTEM IN MULTIPLE CONNECTED DOMAINS

VIKTOR A. POLUNIN, ALEXANDRE P. SOLDATOV

Abstract. In this article we obtain a new integral representation of the general solution of the Moisil-Teodorescu system in a multiply connected domain. Also we give applications of this representation to Riemann-Hilbert problem.

1. Introduction

Consider the Moisil-Teodorescu system [2]

$$M\Big(\frac{\partial}{\partial x}\Big)u(x) = 0, \quad M(\zeta) = \begin{pmatrix} 0 & \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_1 & 0 & -\zeta_3 & \zeta_2 \\ \zeta_2 & \zeta_3 & 0 & -\zeta_1 \\ \zeta_3 & -\zeta_2 & \zeta_1 & 0 \end{pmatrix}, \tag{1.1}$$

for a vector $u(x) = (u_1, u_2, u_3, u_4)$. The identity $M^\top(\zeta)M(\zeta) = |\zeta|^2$ shows that the components of this vector are harmonic functions. Note also that using the notation

$$u = (u_1, v) \tag{1.2}$$

system (1.1) can be written in the form

$$\operatorname{div} v = 0, \quad \operatorname{rot} v + \operatorname{grad} u_1 = 0. \tag{1.3}$$

It is well known [2] that the matrix-valued function $M^\top(x)/|x|^3$, where $\top$ stands for the transposed, is the fundamental solution of the differential operator $M(D)$. Thus the Cauchy type integral

$$(I\psi)(x) = \frac{1}{2\pi}\int_\Gamma \frac{M^\top(y-x)}{|y-x|^3} M[n(y)]\psi(y)d_2y, \quad x \notin \Gamma, \tag{1.4}$$

where Γ is a closed smooth surface and $n(y)$ is a unit normal, defines a solution of (1.1).

Let Γ be a boundary of a finite domain D for which n is an exterior normal, $D' = \mathbb{R}^3 \setminus D$ be an open set and for consistency the notation $D^+ = D$, $D^- = D'$ are

2010 *Mathematics Subject Classification.* 35J46, 45B05, 45E05.
Key words and phrases. Riemann-Gilbert problem; elliptic system; integral of Cauchy type; Fredholm integral operator; Moisil-Teodorescu system.

Submitted August 14, 2016. Published December 5, 2016.

introduced. Let Γ be a Lyapunov surface, $u = I\psi$ and suppose that the function ψ satisfies the Holder condition. Then there exist the limit values

$$u^{\pm}(y_0) = \lim_{x\to y_0, x\in D^{\pm}} u(x), \quad y_0 \in \Gamma,$$

and the analogue of Plemelj-Sokhotskyii formula

$$u^{\pm} = \pm\psi + u^*. \tag{1.5}$$

holds. Here $u^* = I^*\psi$ is defined by the singular integral

$$(I^*\psi)(y_0) = \frac{1}{2\pi}\int_{\Gamma} \frac{M^{\top}(y-y_0)}{|y-y_0|^3} M[n(y)]\psi(y)d_2y.$$

These formulas were first obtained by Bitsadze [3]. Based on the minimal requirements on the smoothness of the surface this result made precise in [7]: if Γ belongs to the class $C^{1,\nu}$, $0 < \nu < 1$, then the operator I is bounded $C^{\mu}(\Gamma) \to C^{\mu}(\overline{D})$, $0 < \mu < \nu$.

Let the matrix-valued function

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{pmatrix}$$

be continuous on Γ and be of the rank 2 at any point $y \in \Gamma$. We consider the following analogue of the Riemann-Hilbert boundary value problem

$$Bu^+ = f, \tag{1.6}$$

for the system (1.1). A natural approach for the study of this problem (in case of special matrices B) using the Cauchy type integrals was proposed by Bitsadze[4]. A complete study of problem (1.6) for the domains homeomorphic to the ball was done by Shevchenko [10, 11]. Another approach based on the integral representation of special type was described in [8, 9].

In this article, we consider the case of arbitrary multiply connected domain. Taking into account a general elliptic theory [5, 1], problem (1.6) is Fredholm one under a so called complementarity condition. This condition can be defined as follows [11, 9]. Consider the vector $s = (s_1, s_2, s_3)$ with components

$$s_1 = b^{12} + b^{34}, \quad s_2 = b^{13} - b^{24}, \quad s_3 = b^{14} + b^{23},$$

where $b^{kj} = b_{1k}b_{2j} - b_{1j}b_{2k}$ are the corresponding minors of the matrix B. Then complementarity condition can be expressed in the form

$$s(y)n(y) \neq 0, \quad y \in \Gamma. \tag{1.7}$$

As shown in [11], if Γ is homeomorphic to a ball, then under the above condition the operator R has a Fredholm property and its index equals to -1. In the case of a arbitrary multiply connected domain D only the Fredholm property of this problem can be stated.

Theorem 1.1. *Suppose the surface Γ belongs to the class $C^{1,\nu}$ and the matrix-valued function $B \in C^{\nu}(\Gamma)$ satisfies (1.7). Then the operator $R : C^{\mu}(\Gamma) \to C^{\mu}(\overline{D})$ of the problem (1.1), (1.6) has a Fredholm property.*

Proof. Every two-component vector $\varphi = (\varphi_1, \varphi_2)$ corresponds to a four-component vector $\psi = \widetilde{\varphi}$ by the formula $\widetilde{\varphi} = (\varphi_1, n\varphi_2)$ and we put

$$(I_0\varphi)(x) = (I\widetilde{\varphi})(x), \quad x \in D. \tag{1.8}$$

So, the operator I_0 acts from the space $C^\mu(\overline{D})$ of two-component vector- valued functions to the space $C^\mu(\overline{D})$ of solutions of the system (1.1) in the domain D. We first prove that this operator has a Fredholm property.

For this purpose we consider the special case of problem (1.6), which is defined by the boundary value condition

$$Cu^+ = f \tag{1.9}$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_1 & n_2 & n_3 \end{pmatrix}.$$

We verify, that the kernel of this problem has a finite dimension.

Indeed if $Cu^+ = 0$ then using notation (1.2) we have

$$u_1^+ = 0, \quad v^+ n = 0. \tag{1.10}$$

Since the function u_1 is harmonic in the domain D, then $u_1 = 0$ and the second equality (1.3) becomes $\operatorname{rot} v = 0$. Hence, in each simply-connected subdomain $D_0 \subseteq D$ the function v can be defined as $\operatorname{grad} w_0$ of some function w_0, which is harmonic by virtue of the first equality of (1.3). If D_1 denotes another simply-connected subdomain D with corresponding representation $v = \operatorname{grad} w_1$, then $w_0 - w_1$ is a locally constant function on the open set $D_0 \cap D_1$ because its gradient equals 0. In the whole multiple-connected domain D, the harmonic function w such that $v = \operatorname{grad} w$ is a multi-valued function. It follows from the second equality of (1.10) that

$$\frac{\partial w^+}{\partial n} = 0. \tag{1.11}$$

To avoid the multipli-connectedness of the domain D let us consider its cuts. By definition the cut is a simply-connected smooth surface $R \subseteq \overline{D}$ with smooth boundary ∂L, such that $R \cap \Gamma = \partial L$. There exist disjoint cuts $R_1, \dots, R_m$, such that the set

$$D_R = D \setminus R, \quad R = R_1 \cup \dots \cup R_m,$$

is a simply connected domain. In this domain the function w is a single-valued and its boundary values satisfy the relation

$$(w^+ - w^-)\big|_{R_i} = c_i, \quad 1 \le i \le m, \tag{1.12}$$

with some constants c_i. Nevertheless equalities $c_1 = \dots = c_m = 0$ indicate that w is univalent function. So it is harmonic in the whole domain D, while in a view of (1.11) this is possible only if w is constant. These arguments prove that the space of solutions of homogeneous problem (1.9) is finite dimensional space.

We denote the operator of the problem (1.9) by S and consider the composition SI_0, which is acting within the space of two-component vector-functions in the space $C^\mu(\Gamma)$. Note that the product $CC^\top$ is the unit 2×2-matrix. Also note that (1.6) can be written as $\widetilde{\varphi} = C^\top \varphi$. So by virtue of (1.4), (1.5) we have the equality $SI_0 = 1 + K_0$ with integral operator K_0, according to the formula

$$(K_0\varphi)(y_0) = \frac{1}{2\pi} \int_\Gamma \frac{k_0(y_0, y)}{|y - y_0|^2} \varphi(y) d_2 y, \quad y_0 \in \Gamma, \tag{1.13}$$

with the matrix-valued function

$$k_0(y_0, y) = C(y_0) M^\top(\xi) M[n(y)] C^\top(y), \quad \xi = \frac{y - y_0}{|y - y_0|}.$$

It is easy to see that

$$M^\top(\xi)M(n)C^\top = \begin{pmatrix} n\xi & 0 \\ [n,\xi]_1 & \xi_1 \\ [n,\xi]_2 & \xi_2 \\ [n,\xi]_3 & \xi_3 \end{pmatrix}, \tag{1.14}$$

where in the sequel brackets denote the vector product, a product without brackets is a scalar product, and $[n,\xi]_k$ are components of the vector $[n,\xi]$. Therefore we get the following expression in explicit form

$$k_0(y_0,y) = \begin{pmatrix} n(y)\xi & 0 \\ n(y_0)[n(y),\xi] & n(y_0)\xi \end{pmatrix} = \begin{pmatrix} n(y)\xi & 0 \\ [n(y_0),n(y)]\xi & n(y_0)\xi \end{pmatrix}.$$

It was stated in [7] under assumption $\Gamma \in C^{1,\nu}$ that the function $k_0(t_0,t)$ belongs to $C^\nu(\Gamma\times\Gamma)$) and equals zero at $t=t_0$. So a kernel of the operator K_0 has weak singularity, and the proper operator is compact in the space $C^\mu(\Gamma)$. According to Riesz theorem we conclude that the image $\operatorname{im}(SI_0)$ is closed subspace of finite co-dimension. Since $\operatorname{im} S \supseteq \operatorname{im}(SI_0)$, then an image of the operator S has the same property. Therefore the operator S has the Fredholm property, and taking into account the Fredholm property of the product $SI_0 = 1 + K$ this implies that I_0 is Fredholm operator. □

Let us next turn to the original problem (1.6). As before it is obvious that $RI_0 = G + K$ with the matrix -valued function $G = BC^\top$ and the integral operator K defined similar (1.13) with respect to the function $k(y_0,y) = B(y_0)M^\top(\xi)M[n(y)]C^\top(y)$, contrary to the previous case, this operator is singular operator.

Since I_0 is Fredholm operator, the operator R of our problem is Fredholm equivalent to the operator $N = G + K$. For a surface Γ, homeomorphic to a ball, the inequality (1.7) provides the Fredholm property of the singular operator N. Since the Fredholm criterion for this operator has local property [6] the similar result is true for arbitrary surface also, and this completes the proof.

Note that the expressions for the matrices $G(y_0)$ and $k(y_0,y)$ can be simplified. To see this we write the matrix $B = (B_{ij})$ in the form

$$B = \begin{pmatrix} B_{11} & b_1 \\ B_{21} & b_2 \end{pmatrix}$$

with the vectors $b_k = (B_{k2}, B_{k3}, B_{k4})$. Then taking into account (1.14) we obtain

$$G = \begin{pmatrix} B_{11} & b_1 n \\ B_{21} & b_2 n \end{pmatrix},$$

$$\begin{aligned} k(y_0,y) &= \begin{pmatrix} B_{11}(y_0)n(y)\xi + b_1(y_0)[n(y),\xi] & b_1(y)\xi \\ B_{21}(y_0)n(y)\xi + b_2(y_0)[n(y),\xi] & b_2(y),\xi \end{pmatrix} \\ &= \begin{pmatrix} B_{11}(y_0)n(y)\xi + [b_1(y_0),n(y)]\xi & b_1(y)\xi \\ B_{21}(y_0)n(y)\xi + [b_2(y_0),n(y)]\xi & b_2(y)\xi \end{pmatrix} \end{aligned}$$

$$\xi = \frac{y-y_0}{|y-y_0|}.$$

References

[1] Agmon, S.; Douglis, A.; Nirenberg, L.; *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II*, Comm. Pure Appl. Math. ,1964, 17, 35-92.

[2] Bitsadze, A. V.; *Foundation of the theory of analytic functions of a complex variable*, Nauka, 1972 (in Russian).
[3] Bitsadze, A. V.; *Spatial analogue of an integral of Cauchy type and some of its applications*, Izv. Akad. Nauk SSSR Ser. Mat., 17:6 (1953), 525538 (in Russian)
[4] Bitsadze, A. V.; *On two-dimensional integral of Cauchy type*, Soob. Akad. nauk Gruz. SSR, 1955, 16, 3, 177-184 in (Russian)
[5] Gilbarg, D.; Trudinger, N. S.; *Elliptic Partial Differential Equations of Second Order Springer*, 1989 - p. 464.
[6] Mikhlin, S. G.; *Multidimensional singular integrals and integral equations.*, Fizmatgiz, 1962 (in Russian)
[7] Polunin, V. A.; Soldatov, A. P.; *Three-dimensional analogue of the Cauchy type integral*, Differential Equations, 2011, vol. 47. No.3, pp. 1-10 DOI: 10.1134/S0012266111030074
[8] Polunin, V. A.; Soldatov, A. P.; *The Riemann- Hilbert problem for Moisil-Theodorescu system in bounded domains*, Non-classical equations of mathematical physics. - Novosibirsk: 2010 pp. 192-201 (in Russian)
[9] Polunin, V. A.; Soldatov, A. P.; *On the integral representation of solutions for Moisil-Theodorescu*, Vedomosty. BSU, 2011., 5(118). pp. 84-92. 2011 (in Russian)
[10] Shevchenko, V. I.; *On the Riemann - Hilbert problem for a holomorphic vector*, Doklady Akad. Nauk SSSR, 1966. Vol. 169, 6. pp. 1285-1288 (in Russian)
[11] Shevchenko, V. I.; *Some boundary value problems for holomorphic vector*, Sborn Math. Phys. Kiev. - 1970. Vol. 8. - pp. 172-187 (in Russian)

Viktor A. Polunin
Belgorod State University, Belgorod, Russia
E-mail address: polunin@bsu.edu.ru

Alexandre P. Soldatov
Belgorod State University, Belgorod, Russia
E-mail address: soldatov48@gmail.com