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# USE OF FINITE DIFFERENCE METHOD FOR NUMERICAL SOLUTION OF THREE-DIMENSIONAL HEAT TRANSFER FRACTIONAL DIFFERENTIAL EQUATION

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**Abstract:** The paper proposes a numerical solution for the mixed problem concerning a three-dimensional heat transfer fractional differential equation, based on the finite difference method. To solve this problem, an explicit difference scheme described in the paper is used. The stability of a proposed difference scheme is proved. The case of homogeneous medium and a square grid is considered.

Keywords:fractional differential equations; heat and mass transfer equation; numerical methods; approximation with fractional derivatives.

## I. INTRODUCTION

The development of the fractional integration and differentiation apparatus technique is of interest both in terms of the development of the fractional integrodifferentiation theory itself and for solving practical problems.

Fractional integrals and derivatives and fractional integrodifferential equations find their application in theoretical physics, mechanics, applied mathematics, where their application allows a deeper understanding of known results and obtaining a new class of solutions, and cover a wide range of problems not previously explained from the standpoint of traditional approaches. Fractional integrodifferentiation is a powerful tool for describing physical systems with memory and non locality characteristics.

Despite the long history of the development of the fractional differentiation mathematical apparatus technique, analytical methods for solving fractional derivative equations have been obtained only for a narrow circle of problems [1–5]. At the same time, numerical methods for solving such problems are being actively developed now [6–8].

## II. METHOD

### Formulation of the problem

Parabolic equations with fractional derivatives [9-11] are used in the study of heat transfer problems. We consider a non stationary three-dimensional heat transfer equation with fractional derivatives.

$$D_t^{\alpha}T(t,x,y,z)=C_xD_x^{\beta}T(t,x,y,z)+C_yD_y^{\beta}T(t,x,y,z)+C_zD_z^{\beta}T(t,x,y,z),$$

Where $C_x$ , $C_y$, $C_z$ are heat conductivity coefficients, $0\langle\alpha\le 1$ , $1\langle\beta\le 2$,

We find a solution to this equation $T(t,x,y,z)\in C^3(D)$ $(D=\{(x,y,z,t):0\le x\le a, 0\le y\le b, 0\le z\le c, 0\le t\le T\})$ satisfying the initial and boundary conditions:

$$T(0,x,y,z)=\psi(x,y,z), x,y,z\in G \qquad (2)$$

$$T(t,x,y,z)\big|_{\Gamma}=\varphi(x,y,z), x,y,z\in G \qquad (3)$$

Where is a parallelepiped with the border $\Gamma$.

The equation contains partial fractional derivatives of Riemann-Liouville:

$$D_x^{\beta}T(t,x,y,z)=\left(\frac{\partial}{\partial x}\right)^2 I^{2-\beta}T(t,x,y,z)$$

and the partial fractional integral of Riemann-Liouville with respect to the corresponding variables:

$$I^{\beta}T(t,x,y,z)=\frac{1}{\Gamma(\beta)}\int_0^x\frac{T(t,s,y,z)}{(x-s)^{1-\beta}}ds.$$

## III. RESULT AND DISCUSSION

We describe the numerical scheme used to solve the equation and introduce a grid in the region D:

We divide the region $[0;a]$ to $L+1$ segments with nodes $x_l=l\Delta x,(l=0,...,L)$ where $\Delta x=\frac{a}{L}$ is a step along $x$ . Then we divide the region $[0;b]$ to $M+1$ segments with nodes $y_m=m\Delta y,(m=0,...,M)$ where $\Delta y=\frac{b}{M}$ is a step along $y$ . Then we divide the region $[0;c]$ to $N+1$ segments with nodes $z_n=n\Delta z,(n=0,...,N)$ where $\Delta z=\frac{c}{N}$ is a step along $z$ . Split the time interval $[0;T]$ into parts with a step $\Delta t$ and time nodes $t_k=k\Delta t,(k=0,...,K)$ .

For convenience, we denote $T_{k,l,m,n}=T(t_k,x_l,y_m,z_n)$ and use the Grünwald formula [ 12 ] to approximate the fractional Riemann – Liouville derivative.

It is known [12, p. 280] that for any function $f(t)$ which allows decomposition in power series, a fractional derivative of order $1-\gamma$ at any point of convergence of this series can be written as a Grünwald-Letnikov derivative:

$$D_t^{1-\gamma}f(t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t^{1-\gamma}}\sum_{k=0}^{\left[\frac{t}{\Delta t}\right]}\omega_k^{(1-\gamma)}f(t-k\Delta t) \qquad (4)$$

Where $\omega_0=1$, $\omega_k=\frac{(-1)^k\beta(\beta-1)...(\beta-k+1)}{k!}$

If the function is continuous and its derivative is integrable over the segment $[0;t]$ , then Riemann-Liouville and Grünwald-Letnikov derivatives of any order $0<1-\gamma<1$ exist and match for any point in time from $[0;t]$ [ 9 ].

The Grunwald-Letnikov definition allows us to numerically find the Riemann-Liouville derivative:

$$D_t^{1-\gamma}f(t)\approx\frac{1}{\Delta t^{1-\gamma}}\sum_{k=0}^{\left[\frac{t}{\Delta t}\right]}\omega_k^{(1-\gamma)}f(t-k\Delta t) \qquad (5)$$

Using the formula (5) for the fractional Riemann-Liouville derivatives with respect to the spatial variables, we obtain:

$$D_x^{\beta}T(t,x,y,z)\big|_L=\frac{1}{\Delta x^{\beta}}\sum_{j=0}^{L+1}\omega_j^{\beta}T(t,x-j\Delta x,y,z)$$

$$D_y^{\beta}T(t,x,y,z)\big|_M=\frac{1}{\Delta y^{\beta}}\sum_{j=0}^{M+1}\omega_j^{\beta}T(t,x,y-j\Delta y,z),$$

$$D_z^{\beta}T(t,x,y,z)\big|_N=\frac{1}{\Delta z^{\beta}}\sum_{j=0}^{N+1}\omega_j^{\beta}T(t,x,y,z-j\Delta z).$$

Considering that $x_{L-j}=x_L-j\Delta x,\ y_{M-j}=y_M-j\Delta y,$ $z_{N-j}=z_N-j\Delta z,$ we will present Riemann-Liouville derivative at $0\langle\alpha\le 1$ in the form [6]:

$$D_{0t}^{\alpha}T(t,x,y,z)\,|_{tk}=\frac{1}{\Gamma(1-\alpha)}\left(\frac{T(t_k,x,y,z)}{(t_{k+1}-t_k)^{\alpha}}+\int_{tk}^{tk+1}\frac{T'(s,x,y,z)ds}{(t_{k+1}-s)^{\alpha}}\right)$$

(6)

Where $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$

We present the derivative $\dfrac{dT(\mathrm{s,x,y,z})}{ds}$ on the segment $[t_k;t_{k+1}]$ in the form of a finite difference [13, p.47]:

$$\left(\frac{dT}{ds}\right)_k \approx \frac{T(t_{k+1},x,y,z)-T(t_k,x,y,z)}{\Delta t} \qquad (7)$$

In view of (6), the Riemann-Liouville derivative of fractional order $[t_k;t_{k+1}]$ at the interval can be approximated by the finite difference:

$$D_{0t}^{\alpha}T(t,x,y,z)|_{tm}=\frac{1}{\Gamma(1-\alpha)}\left(\frac{T(t_m,x,y,z)}{\Delta t^{\alpha}}+\frac{T(t_{m+1},x,y,z)-T(t_m,x,y,z)}{\Delta t}\int_{tm}^{tm+1}\frac{ds}{(t_{m+1}-s)^{\alpha}}\right)=$$
$$\frac{T(t_{m+1},x,y,z)-\alpha T(t_m,x,y,z)}{\Gamma(1-\alpha)(1-\alpha)\Delta t^{\alpha}}$$

As a result, for the heat transfer equation (1), we write the explicit difference scheme:

$$\frac{T_{k+1,l,m,n}-\alpha T_{k,l,m,n}}{\Gamma(1-\alpha)(1-\alpha)\Delta t^{\alpha}}=\frac{C_x}{\Delta x^{\beta}}\sum_{j=0}^{l+1}\omega_j^{\beta}T_{k,l-j+1,m,n}+\frac{C_y}{\Delta y^{\beta}}\sum_{j=0}^{m+1}\omega_j^{\beta}T_{k,l,m-j+1,n}+\frac{C_z}{\Delta z^{\beta}}\sum_{j=0}^{n+1}\omega_j^{\beta}T_{k,l,m,n-j+1}$$

or taking into account the values $\omega_j$

$$\begin{aligned}\frac{T_{k+1,l,m,n}-\alpha T_{k,l,m,n}}{\Gamma(1-\alpha)(1-\alpha)\Delta t^{\alpha}}&=\frac{C_x}{\Delta x^{\beta}}(T_{k,l+1,m,n}-\beta T_{k,l,m,n}+\sum_{j=2}^{l+1}\omega_j^{\beta}T_{k,l-j+1,m,n})+\\&\frac{C_y}{\Delta y^{\beta}}(T_{k,l,m+1,n}-\beta T_{k,l,m,n}+\sum_{j=2}^{m+1}\omega_j^{\beta}T_{k,l,m-j+1,n})+\\&\frac{C_z}{\Delta z^{\beta}}(T_{k,l,m,n+1}-\beta T_{k,l,m,n}+\sum_{j=2}^{n+1}\omega_j^{\beta}T_{k,l,m,n-j+1})\end{aligned}$$

The decision $T(t_{k+1},x,y,z)$ can be represented as:

$$T(t_{k+1},x,y,z)=ST(t_k,x,y,z),$$

Where S is the operator for a transition from onetime layer to another.
Then we find out the stability of the scheme. We have

$$\|T(t_{k+1},x,y,z)\|=\|ST(t_k,x,y,z)\|\le\|S\|\cdot\|T(t_k,x,y,z)\| \text{ and}$$
$$\|T(t_{k+1},x,y,z)\|\le\|T(t_k,x,y,z)\|\le\ldots\le\|T(t_0,x,y,z)\|$$

From where it follows, that the initial disturbances fade out. Given the initial data, the stability condition of the difference scheme (8) is [14]. This means that the spectrum of the operator lies inside a circle of a unit radius on the complex plane. From there, , where is any eigenvalue of the transition operator. We find the eigenvalues of the transition operator, for this, we will present the solution $T(t_k,x,y,z)$ in the form of disturbance.

$$T(t_k,x,y,z)=\lambda^k\exp(i\omega_x l\Delta x)\exp(i\omega_y m\Delta y)\exp(i\omega_z n\Delta z), \qquad (9)$$

Where $i$ is an imaginary unit, and $\lambda$ is an eigenvalue of the transition operator [14].

We substitute equality (9) into (8), and then obtain the estimate for the eigenvalues of the transition operator:

$$\begin{aligned}\lambda\le\alpha&-\frac{\Gamma(2-\alpha)C_x\Delta t^{\alpha}}{\Delta x^{\beta}}(4\sin^2\frac{\omega_x\Delta x}{2}+\beta-2)-\\&-\frac{\Gamma(2-\alpha)C_y\Delta t^{\alpha}}{\Delta y^{\beta}}(4\sin^2\frac{\omega_y\Delta y}{2}+\beta-2)-\\&-\frac{\Gamma(2-\alpha)C_z\Delta t^{\alpha}}{\Delta z^{\beta}}(4\sin^2\frac{\omega_z\Delta z}{2}+\beta-2).\end{aligned} \qquad (10)$$

From where we have, that all the eigenvalues of the transition operator do not exceed 1 in absolute value. We get:

$$0\le\frac{\Gamma(2-\alpha)C_x\Delta t^{\alpha}}{\Delta x^{\beta}}+\frac{\Gamma(2-\alpha)C_y\Delta t^{\alpha}}{\Delta y^{\beta}}+\frac{\Gamma(2-\alpha)C_z\Delta t^{\alpha}}{\Delta z^{\beta}}\le\frac{\alpha+1}{\beta+2}.$$

## IV. CONCLUSION

Thus, the following assertion is proved.Statement 1.
Explicit difference scheme

$$\frac{T_{k+1,l,m,n}-\alpha T_{k,l,m,n}}{\Gamma(1-\alpha)(1-\alpha)\Delta t^{\alpha}}=\frac{C_x}{\Delta x^{\beta}}\sum_{j=0}^{l+1}\omega_j^{\beta}T_{k,l-j+1,m,n}+\frac{C_y}{\Delta y^{\beta}}\sum_{j=0}^{m+1}\omega_j^{\beta}T_{k,l,m-j+1,n}+\frac{C_z}{\Delta z^{\beta}}\sum_{j=0}^{n+1}\omega_j^{\beta}T_{k,l,m,n-j+1}$$

Is stable if

$$\Delta t^{\alpha}\left(\frac{C_x}{\Delta x^{\beta}}+\frac{C_y}{\Delta y^{\beta}}+\frac{C_z}{\Delta z^{\beta}}\right)\le\frac{\alpha+1}{(\beta+2)\Gamma(2-\alpha)}.$$

We consider the case of a homogeneous medium $C_x=C_y=C_z=C$ and a square grid when $\Delta x=\Delta y=\Delta z$ . The difference scheme (8) will be:

$$\begin{aligned}\frac{T_{k+1,l,m,n}-\alpha T_{k,l,m,n}}{\Gamma(1-\alpha)(1-\alpha)\Delta t^{\alpha}}=\frac{C}{\Delta x^{\beta}}\Big(&(T_{k,l+1,m,n}+\sum_{j=2}^{l+1}\omega_j^{\beta}T_{k,l-j+1,m,n})+\\&(T_{k,l,m+1,n}+\sum_{j=2}^{m+1}\omega_j^{\beta}T_{k,l,m-j+1,n})+\\&(T_{k,l,m,n+1}+\sum_{j=2}^{n+1}\omega_j^{\beta}T_{k,l,m,n-j+1})-3\beta T_{k,l,m,n}\Big)\end{aligned} \qquad (12)$$

An explicit difference scheme (12) is stable when:

$$\Delta t^{\alpha}\le\frac{\Delta x^{\beta}\ (\alpha+1)}{3C(\beta+2)\Gamma(2-\alpha)}.$$

A great difficulty is inherent in a computational experiment based on the obtained difference scheme due to an increase in the volume of computations as the value of the time variable increases. As a result, the computation time of even a one-dimensional problem on a relatively coarse grid takes several hours. Therefore, for such calculations, it seems reasonable to use multiprocessor computing, which entails the need to appropriately adapt existing methods and algorithms.

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