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Quadratic exponential in modified discrete Fourier transform and shifted Gaussian series

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Abstract. In this paper we consider quadratic exponential in two main constructions: as finite Gauss sums and in connection with modified discrete Fourier transforms and also as basis of series in shifted Gaussians. We study spectral properties of modified Fourier discrete transforms, its eigenspaces and eigenvectors. For the case of $n = 4k$ an effect of broken symmetry for discrete Fourier transform is discussed. An application of modified discrete Fourier transform to cryptography is proposed. Also series in quadratic exponentials are considered. An application of series in shifted Gaussians to interpolation problem is analyzed. The correctness of finite system approximation is proved and results of good approximations are illustrated by graphs.

1. Introduction

In this paper we consider some problems in which instead of classical linear exponential

$$L_1(x, y) = \exp(axy)$$

quadratic exponential

$$Q_2(x, y) = \exp(ax^2 + by^2 + cxy) \tag{1}$$

works and gives important consequences and results. Quadratic exponential is applied in such different problems and fields as Fresnel waves, Gabor frames, holography, GAUSSIAN computer package and more. First idea is to use quadratic exponential instead of linear ones was applied by Fresnel to understand the results of Fraunhofer [1, 17]. He also introduced famous Fresnel integrals in connection with this topic.

In the first section we consider quadratic exponential in finite sums. This is a famous Gauss sum which is a direct generalization of geometric progression

$$L_n(x) = \sum_{k=0}^{n} \exp(ixk) \tag{2}$$

for the case of quadratic exponential

$$Q_n(x) = \sum_{k=0}^{n} \exp(ixk^2). \tag{3}$$

We consider a problem of spectral properties of Discrete Fourier Transform (DFT) matrix for which Gauss sum is a trace value. DFT matrix is defined by

$$f_{kj} = \frac{1}{\sqrt{n}} \exp(-i\frac{2\pi kj}{n}),\ 0 \leq k \leq n-1,\ 0 \leq j \leq n-1. \quad (4)$$

For continuous case all eigenspaces are completely symmetrical but for the discrete case the symmetry is suddenly broken. To improve this we introduce modified DFT (MDFT), which are defined based on permutations of complex roots of unity, namely

$$F_r = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ r_1^2 & r_2^2 & \dots & r_n^2 \\ \dots & \dots & & \dots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{pmatrix},$$

where $r_1^{k-1}, r_2^{k-1}, \dots, r_n^{k-1}$ are roots of unity. We study for MDFT its spectra, eigenvectors and dimensions of eigenspaces, cf. [22].

Applications of MDFT to cryptography are considered briefly.

In the second section we consider application of quadratic exponentials to interpolation problems. We interpolate using integer shifts of Gaussians that is quadratic exponentials again. This system is interesting as it is incomplete and not orthogonal but demonstrate good approximating properties [12, 28].

The basic interpolating problem is:

$$g(x) = \sum_{k=-\infty}^{k=\infty} f_k \exp\left(-\frac{(x-k)^2}{2s^2}\right), \quad (5)$$

$$g(m) = f(m), m \in \mathbb{Z}, \quad (6)$$

where $g(x)$ is an approximation in the form of the series of Gaussians and coincides with a given function $f(x)$ at integers.

Different approaches for this problem meet enough problems, cf. [28, 23, 24, 9]. Note that usage of Jacobi theta–functions [3, 26] is essential in this kind of problems.

The paper is mostly a survey of known results and also of the results proved by authors but some results are new and some with new proofs.

2. Quadratic exponentials in finite Gauss sums and modified discrete Fourier transforms

As we mentioned a generalization of geometric progression

$$L_n(x) = \sum_{k=0}^{n} \exp(ixk) \quad (7)$$

for integer or rational x is the famous Gauss sum

$$Q_n(x) = \sum_{k=0}^{n} \exp(ixk^2). \quad (8)$$

Squared module of the sum $Q_n(x)$ and consequently the module itself is easily calculated, so to find the sum $Q_n(x)$ it is enough to determine just a sigh in

$$Q_n(x) = \pm |Q_n(x)|. \quad (9)$$

Suddenly for himself Gauss met serious difficulties and found the correct formula for this mysterious sigh for 11 years, the result is for $n \in \mathbb{N}$

$$S_n = \sum_{k=0}^{n-1} \exp(2\pi i \frac{k^2}{n}) = 1/2(1+i)(1+i^{-n})\sqrt{n}. \tag{10}$$

Note that for the cubic Gauss sum

$$S_n = \sum_{k=0}^{n-1} \exp(2\pi i \frac{k^3}{n}) \tag{11}$$

a formula is still unknown [5].

Now let us consider discrete Fourier transform (DFT) which is one of the most known and useful mathematical instruments. DFT is successfully applied in electrodynamics and optics, for codes and cryptography, for analysing communication systems and filtering, algorithms of data compressing and computerized tomography [6, 13, 16]. DFT has direct connections with Gauss sums and Jacobi theta–functions.

For applications DFT is also important as problems of calculations of cyclic convolutions, big numbers products, and polynomes algebra are based on DFT, also on its fast variants, cf [6, 13, 16, 27].

Recover a definition of DFT matrix

$$f_{kj} = \frac{1}{\sqrt{n}} \exp(-i\frac{2\pi kj}{n}), \ 0 \le k \le n-1, \ 0 \le j \le n-1.$$

Say for $n = 4$ it is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

Consider a problem of evaluation of spectrum for this matrix. It is unitary, so eigenvalues are $\pm 1, \pm i$, the only question is in dimensions of eigenspaces. This is presented in the Table 1.

The general result for any n is summarized in the Table 2.

This result was proved by I. Schur in 1921 in [21], the proof is based on evulation of Gauss sums, it was many times reopened.

To cure break of symmetry in eigenspaces dimensions for $n = 4k$ we consider *modified DFT*.

Definition. Consider roots of unity in standard order $r_1, r_2, \ldots, r_n$. Define MDFT as the matrix with arbitrary permuted string $r_1^{k-1}, r_2^{k-1}, \ldots, r_n^{k-1}$

$$F_r = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_n \\ r_1^2 & r_2^2 & \ldots & r_n^2 \\ \ldots & \ldots & & \ldots \\ r_1^{n-1} & r_2^{n-1} & \ldots & r_n^{n-1} \end{pmatrix}.$$

There are $n!$ variants of MDFT for any n. Say for $n = 4$ there are 24 MDFT matrices.

Now we list results of computer calculations of spectral data for the case $n = 4$ for different MDFT matrices.

1) $r = (1, -i, -1, i)$ (standard DFT), 2) $r = (1, i, -1, -i)$ (DFT inverse).

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Table 1. Dimensions of eigenspaces.

n	*1*	*i*	*-1*	*-i*
2	1	0	1	0
3	1	**1+**	1	0
4	2	0	1	**1+**
5	2	**1+**	1	1
6	2	1	**2+**	1
7	2	1	2	**2+**
8	**3+**	1	2	2
9	3	**2+**	2	2
10	3	2	**3+**	2
11	3	2	3	**3+**
12	**4+**	2	3	3
13	4	**3+**	3	3
14	4	3	**4+**	3
15	4	3	4	**4+**
16	**5+**	3	4	4
17	5	**4+**	4	4
18	5	4	**5+**	4
19	5	4	5	**5+**
20	**6+**	4	5	5

Table 2. Formulas for dimensions of eigenspaces.

n	*1*	*i*	*-1*	*-i*
4N	***N+1 (+)***	N-1	N	N
4N+1	N+1	***N (+)***	N	N
4N+2	N+1	N	***N+1 (+)***	N
4N+3	N+1	N	N+1	***N+1 (+)***

Squaring DFT matrix is a permutation one

$$F_r^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

spectrum is not simple $(1, 1, -1, -i)$,

characteristic polynomial: $x^4 - (1-i)x^3 - (1+i)x^2 + (1-i)x + i$,

real eigenvectors: $(-1, 1, 1, 1), (0, -1, 0, 1), (2, 1, 0, 1), (1, 0, 1, 0)$.

3) $r = (-1, i, 1, -i)$ 4) $r = (-1, -i, 1, i)$.

Squaring this MDFT matrix is a permutation one

$$F_r^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

spectrum is not simple $(1, i, i, -i)$,
characteristic polynomial:

$$x^4 - (1+i)x^3 + (1+i)x^2 - (1+i)x + i,$$

complex eigenvectors: $(0, -1, 0, 1), (-1, -2i, 1, 0), (i, 1, -i, 1), (1, 0, 1, 0)$.
In cases 1–4 roots permutations are cyclic and starting by not a primitive root. The 4th power is an identical matrix. They are also diagonizable.
5) $r = (-1, i, 1, -i)$, 6) $r = (-1, -i, 1, i)$, 7) $r = (i, -1, -i, 1)$,
8)$r = (-i, -1, i, 1)$.
In cases 5–8 roots permutations are cyclic and starting by primitive roots $-i, i$.
Squaring:

$$F_r^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \end{pmatrix}, F_r^4 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \qquad F_r^{16} = E.$$

characteristic polynomial: $x^4 - i$.
IMPORTANT: This MDFT has a simple spectrum consisting of $\sqrt[4]{i}$.
MATHEMATICA meet problems from this step in calculations, it gives first eigenvector in the form roots

$$\left\{ s, (1 - \sqrt{2} + \sqrt{4 - 2\sqrt{2}})i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, 1 \right\},$$

via the root s of an equation

$$s^8 - 8s^7 + 32s^6 - 24s^5 + 2s^4 + 24s^3 + 32s^2 + 8s + 1 = 0.$$

9) $r = \{1, -1, i, -i\}$, 10) $r = \{1, -1, -i, i\}$, 11) $r = \{1, i, -i, -1\}$, 12) $r = \{1, -i, i, -1\}$.
Permutations starting by 1.
Squaring:

$$F_r^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{i}{2} & -\frac{i}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \\ 0 & \frac{1}{2} - \frac{i}{2} & \frac{1}{2} & \frac{i}{2} \end{pmatrix}.$$

characteristic polynomial:

$$x^4 + \left(\frac{1}{2} - \frac{i}{2}\right)x^3 - (1+i)x^2 - \left(\frac{1}{2} - \frac{i}{2}\right)x + i.$$

IMPORTANT: This MDFT has a simple spectrum

$$\left\{ -\frac{\sqrt{7}+1}{4} - \frac{\sqrt{7}-1}{4}i, \frac{\sqrt{7}-1}{4} + \frac{\sqrt{7}+1}{4}i, -1, 1 \right\}.$$

eigenvalues:

$$\left\{0, \frac{1}{2}i\left((2+i)+\sqrt{7}\right), -\frac{1}{2}i\left((2-i)+\sqrt{7}\right), 1\right\},$$

$$\left\{0, -\frac{1}{2}i\left((-2-i)+\sqrt{7}\right), \frac{1}{2}i\left((-2+i)+\sqrt{7}\right), 1\right\},$$

$$\{-1,1,1,1\},$$

$$\{3,1,1,1\}.$$

Similar results are received for other MDFT defined by permutations:

$$13)r=\{i,1,-1,-i\}, 14)r=\{-i,1,-1,i\}, 15)r=\{-1,1,i,-i\}, 16)r=\{-1,1,-i,i\},$$

$$17)r=\{i,-1,1,-i\}, 18)r=\{-i,i,1,-1\}, 19)r=\{-i,-1,1,i\}, 20)r=\{i,-i,1,-1\},$$

$$21)r=\{i,-i,-1,1\}, 22)r=\{-i,i,-1,1\}, 23)r=\{-1,i,-i,1\}, 24)r=\{-1,-i,i,1\}.$$

Unfortunately it is very difficult to prove results on MDFT spectral properties analytically and strict, we have only computer calculations. It seems that the next hypotheses is well affirmed, it was proved by computer calculations for $n=4,5$.

Hypothesis. For $n=4k$ only for MDFT based on cycling permutations starting by not primitive roots dimension of eigenspaces are not equal, so we have eigenspaces dimension break of symmetry.

So the standard DFT is the worse one from point of view of its spectral properties and dimension symmetry of eigenspaces.

We consider just rows permutations, but of course it is possible also to consider lines permutations or them both. Also fast versions of MDFT are interesting and important [6, 13, 16]. DFT is essential in cryptography [27], so we may use MDFT instead, it gives $n!$ new variants with no need of further correction of standard DFT algorithms.

A difficult problem is to evaluate a generalized Gauss sum, namely trace of MDFT

$$G(P,Q,n)=\sum_{k=0}^{n-1}\exp\left(i\frac{2\pi}{n}p(k)q(k)\right), P=(p(0),p(1),...,p(n-1)),$$

$$Q=(q(0),q(1),...,q(n-1)),$$

where P,Q — two arbitrary permutations of numbers $(0,1,\ldots,n-1)$.

3. Quadratic exponential in interpolation problem

For many years the main idea for function expansions was to use complete orthogonal systems. But they have serious restrictions mainly connected with instability. So for the last years in different fields of mathematics and applications incomplete, non–orthogonal and overloaded systems are widely applied. Such systems are used in problems of electric and optic signals, filtering, holography, computer modeling in tomography and medicine. The systems in use are different frames, wavelets, Gabor systems or coherent states, Rvachevs functions and so on.

Consider a problem of interpolating of arbitrary function by a series of integer shifts of Gaussians, that quadratic exponentials with parameters. It is known that this system is incomplete in standard spaces, but nevertheless it is very useful and often applied, cf. [12, 28, 23, 24, 9].

Consider the next problem

Interpolation problem: for any function $f(x)$ defined for $x \in \mathbb{R}$ and for some parameter $\sigma > 0$ find a function $\tilde{f}(x), x \in \mathbb{R}$ as a series in Gaussians with integer shifts

$$\tilde{f}(x) \sim \sum_{k=-\infty}^{\infty} f_k e^{-\frac{(x-k)^2}{2\sigma^2}} \tag{12}$$

which coincides with $f(x)$ at integer points

$$f(m) = \tilde{f}(m), \ m \in \mathbb{Z}. \tag{13}$$

Some approaches to solve this interpolation problem are known. The explicit formula via theta–functions was found in [12], but as it was demonstrated in [28] this formula is useless for real calculations due to necessary divisions on very small values. Also DFT was used but with good results in a restricted range of parameters. The direct approach was proposed in [23, 24] to reduce a problem to solving linear systems. Let consider this method in more details.

Let use the next notation

$$e(\sigma, x, k) = e^{-\frac{(x-k)^2}{2\sigma^2}}. \tag{14}$$

A solution is to find coefficients f_k from (12). For it we have to find node functions for integer nodes. In fact we have to find only one node function of the form

$$G(\sigma, x) = \sum_{k=-\infty}^{\infty} g_k \ e(\sigma, x, k). \tag{15}$$

Let us reduce a problem to linear system. From (13) it follows for $m \in \mathbb{Z}$:

$$G(\sigma, m) = \sum_{k=-\infty}^{\infty} g_k \ e(\sigma, m, k) = \sigma_{m0}, \tag{16}$$

σ_{m0} is Kronecker symbol,

$$\sigma_{m0} = \begin{cases} 1, \ m = 0 \\ 0, \ m \neq 0. \end{cases}$$

On finding $G(\sigma, x)$ satisfying (16) we may find also a node function

$$G_l(\sigma, x) = G(\sigma, x - l)$$

for the node $x = l$, as for all m

$$G_l(\sigma, m) = G(\sigma, m - l) = \sigma_{ml}.$$

So the solution will be

$$\tilde{f}(x) = \sum_{l=-\infty}^{\infty} f(l)\, G_l(\sigma, x), \tag{17}$$

as

$$f(m)\, G_m(\sigma, m) = f(m) \cdot 1 = f(m).$$

For going from (17) to (12) let make a change and using (15):

$$\tilde{f}(x) = \sum_{l=-\infty}^{\infty} f(l)\, G_l(\sigma, x) =$$

$$= \sum_{l=-\infty}^{\infty} f(l)\, G(\sigma, x-l) = \sum_{l=-\infty}^{\infty} f(l) \sum_{l=-\infty}^{\infty} g_k\; e(\sigma, x-l, k)$$

Introduce new index $j = l + k$ instead of $l = j - k$, change summing order and then

$$\begin{gathered} \tilde{f}(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(j-k) g_k\; e(\sigma, x-j, k) = \\ = \sum_{l=-\infty}^{\infty} \left\{ \sum_{l=-\infty}^{\infty} f(j-k)\, g_k \right\} e(\sigma, x-j, k) = \sum_{j=-\infty}^{\infty} f_j\; e(\sigma, x, j)\,, \end{gathered} \tag{18}$$

with final formula for coefficients

$$f_k = \sum_{j=-\infty}^{\infty} f(k-j)\, g_j, \tag{19}$$

$f(m)$ is function values at integers, and g_j is coefficients of the node function (15).

Change a system

$$\sum_{k=-\infty}^{\infty} g_k e(\sigma, m, k) = \sigma_{m0},\; m \in \mathbb{Z}.$$

letting $q = e^{-\frac{1}{2\sigma^2}}$. Then

$$\sum_{k=-\infty}^{\infty} g_k q^{(m-k)^2} = \sigma_{m0},\; m \in \mathbb{Z}. \tag{20}$$

For numerical solution we need to consider finite systems cutting (20).

Now let us find finite approximations $H(\sigma, x)$ for node function (15) $G(\sigma, x)$ as

$$H(n, x, \sigma) = \sum_{k=-n}^{n} d_k \cdot q^{(x-k)^2},\; q = \exp\left(-\frac{1}{2\sigma^2}\right),\; 0 < q < 1, \tag{21}$$

and infinite system (20) is reduced to a finite one.

$$H(n, m, j, \sigma) = \delta_{0j},\; j = -m, \ldots, 0, \ldots m,\; m \geq n. \tag{22}$$

From (21)–(22), we have $2m + 1$ equations and $2n + 1$ unknown values d_k, $-n \leq k \leq n$. The matrix form of (21)–(22) is

$$A \cdot d = y, \tag{23}$$

with

$$a_{ij} = q^{(i-j)^2},\; y_j = \delta_{0j},\; i = -n, \ldots, 0, \ldots, n,\; j = -m, \ldots, 0, \ldots, m.$$

Prove that the system (23) is correct, it has the unique solution in case of square matrix. In this we follow [24].

Theorem 1. The matrix A for $m = n$ is not degenerate and its determinant equals

$$|A| = q^{\frac{2n(n+1)(2n+1)}{3}} \cdot W\left(q^{-2n}, \ldots 1, \ldots, q^{2n}\right). \tag{24}$$

Proof. We have

$$|A| = \begin{vmatrix} 1 & q & q^4 & q^9 & q^{16} \\ q & 1 & q & q^4 & q^9 \\ q^4 & q & 1 & q & q^4 \\ q^9 & q^4 & q & 1 & q \\ q^{16} & q^9 & q^4 & q & 1 \end{vmatrix}$$

and

$$a_{ij} = q^{(i-j)^2} = q^{i^2} \cdot q^{-2ij} \cdot q^{j^2}.$$

So, from i–line take q^{i^2} and from j–row take q^{j^2}. Do it for all lines and rows.

$$|A| = q^4 \cdot q \cdot 1 \cdot q \cdot q^4 \cdot \begin{vmatrix} q^{-4} & q^{-3} & 1 & q^5 & q^{12} \\ 1 & q^{-1} & 1 & q^3 & q^8 \\ q^4 & q & 1 & q & q^4 \\ q^8 & q^3 & 1 & q^{-1} & 1 \\ q^{12} & q^5 & 1 & q^{-3} & q^{-4} \end{vmatrix} = q^{20} \cdot \begin{vmatrix} q^{-8} & q^{-4} & 1 & q^4 & q^8 \\ q^{-4} & q^{-2} & 1 & q^2 & q^4 \\ 1 & 1 & 1 & 1 & 1 \\ q^4 & q^2 & 1 & q^{-2} & q^{-4} \\ q^8 & q^4 & 1 & q^{-4} & q^{-8} \end{vmatrix}.$$

After more simplification

$$q^{20} \cdot \begin{vmatrix} q^{-8} & q^{-4} & 1 & q^4 & q^8 \\ q^{-4} & q^{-2} & 1 & q^2 & q^4 \\ 1 & 1 & 1 & 1 & 1 \\ q^4 & q^2 & 1 & q^{-2} & q^{-4} \\ q^8 & q^4 & 1 & q^{-4} & q^{-8} \end{vmatrix} = q^{20} \cdot \begin{vmatrix} 1 & q^4 & q^8 & q^{12} & q^{16} \\ 1 & q^2 & q^4 & q^6 & q^8 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & q^{-2} & q^{-4} & q^{-6} & q^{-8} \\ 1 & q^{-4} & q^{-8} & q^{-12} & q^{-16} \end{vmatrix} =$$

$$= q^{20} \cdot \begin{vmatrix} 1 & q^4 & (q^4)^2 & (q^4)^3 & (q^4)^4 \\ 1 & q^2 & (q^2)^2 & (q^2)^3 & (q^2)^4 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & q^{-2} & (q^{-2})^2 & (q^{-2})^3 & (q^{-2})^4 \\ 1 & q^{-4} & (q^{-4})^2 & (q^{-4})^3 & (q^{-4})^4 \end{vmatrix}.$$

In general case:

$$\det A = \left(\prod_{i=-n}^{n} q^{i^2} \right)^2 \cdot \left(\prod_{j=-n}^{n} q^{j^2} \right)^2 \cdot \prod_{i=-n}^{n} q^{2ni} \cdot W\left(q^{-2n}, \dots 1, \dots, q^{2n}\right) =$$

$$= q^{\frac{2n(n+1)(2n+1)}{3}} \cdot W\left(q^{-2n}, \dots 1, \dots, q^{2n}\right) = q^{\frac{2n(n+1)(2n+1)}{3}} \cdot \prod_{i,j=-n, i \neq j}^{n} \left(q^{-2i} - q^{-2j}\right).$$

Vandermonde det is not zero due to $0 < q < 1$. The theorem is proved.

Remark. The problem to prove correctness for linear system (23) was initiated by L. A. Minin and S. M. Sitnik, determinant formula was found by A. S. Timashov, the strict result was proved by S. N. Ushakov, cf. [24].

Coefficients d_k for node function defined by (21), which are solutions of (22)–(23) may be found explicitly.

Theorem 2. For coefficients d_k the next formula is valid:

$$d_k = (-1)^k q^{-k^2} \frac{W_{k,\, n+1}\left(q^{-2n}, \dots, q^0, \dots, q^{2n}\right)}{W\left(q^{-2n}, \dots, q^0, \dots, q^{2n}\right)}. \tag{25}$$

At the end let us illustrate considered type of approximations and its effectiveness following to the paper [23].

First consider node function close to origin (see figure 1). It is seen that node function obeys its conditions at integers. After that consider some approximations for standard electric signals. The a rectangular signal is presented on the figure 2. The triangular signal is presented on the figure 3. So the method applied gives good approximation for considered signals.

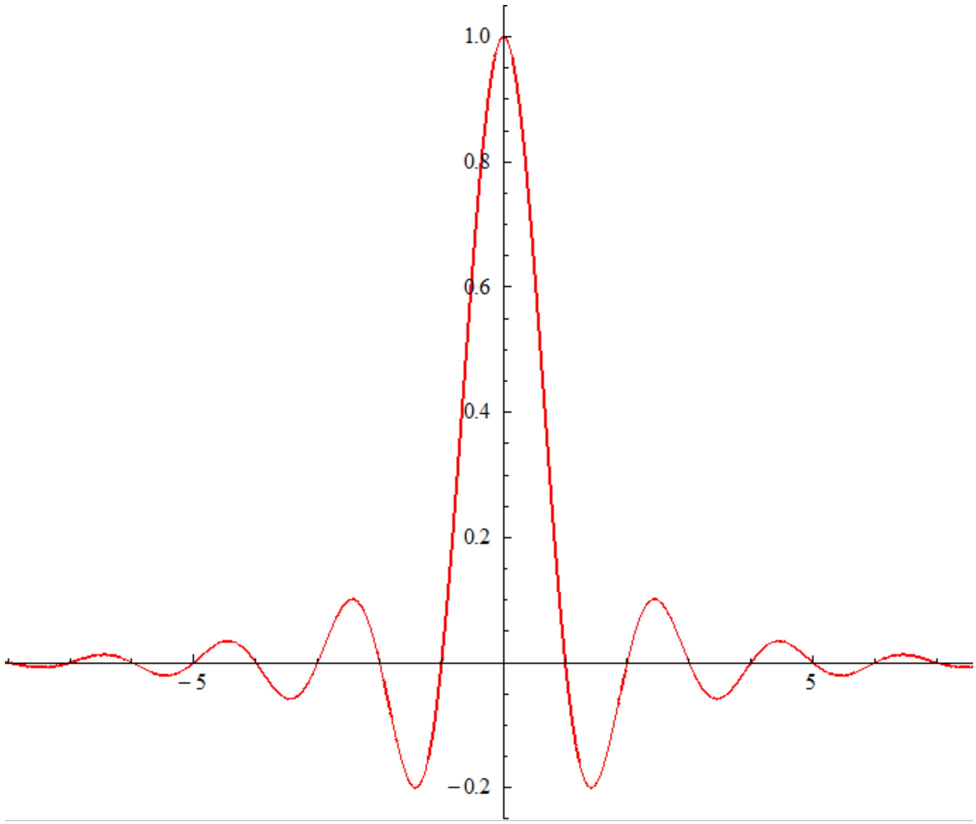

Figure 1. Node function close to origin.

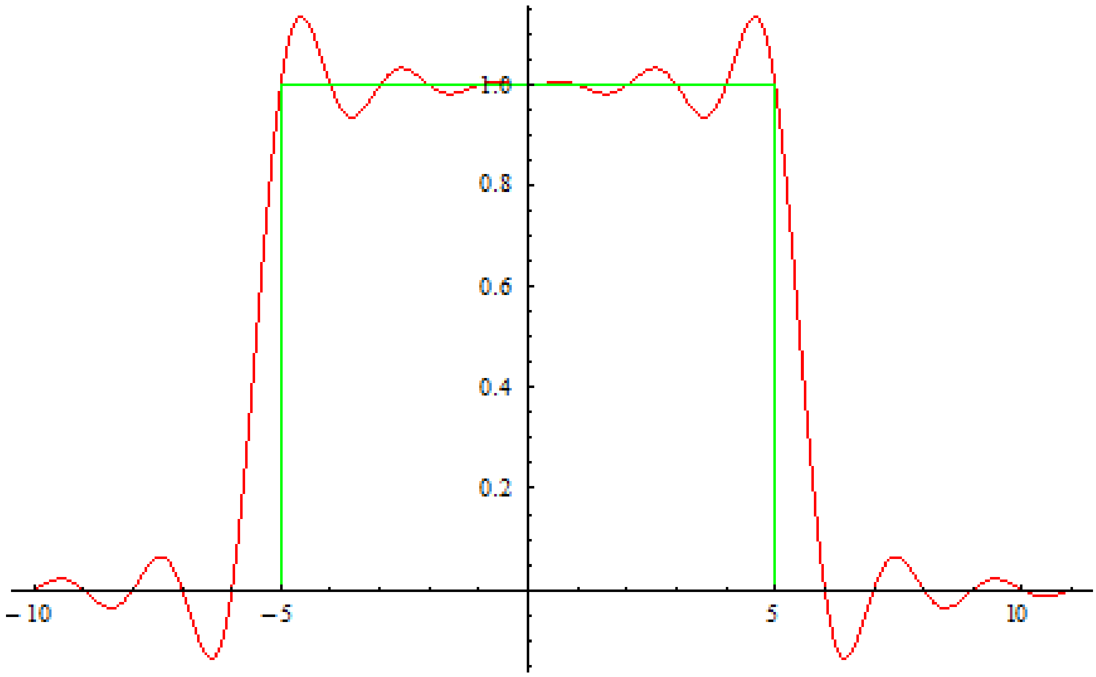

Figure 2. Interpolation of rectangular signal.

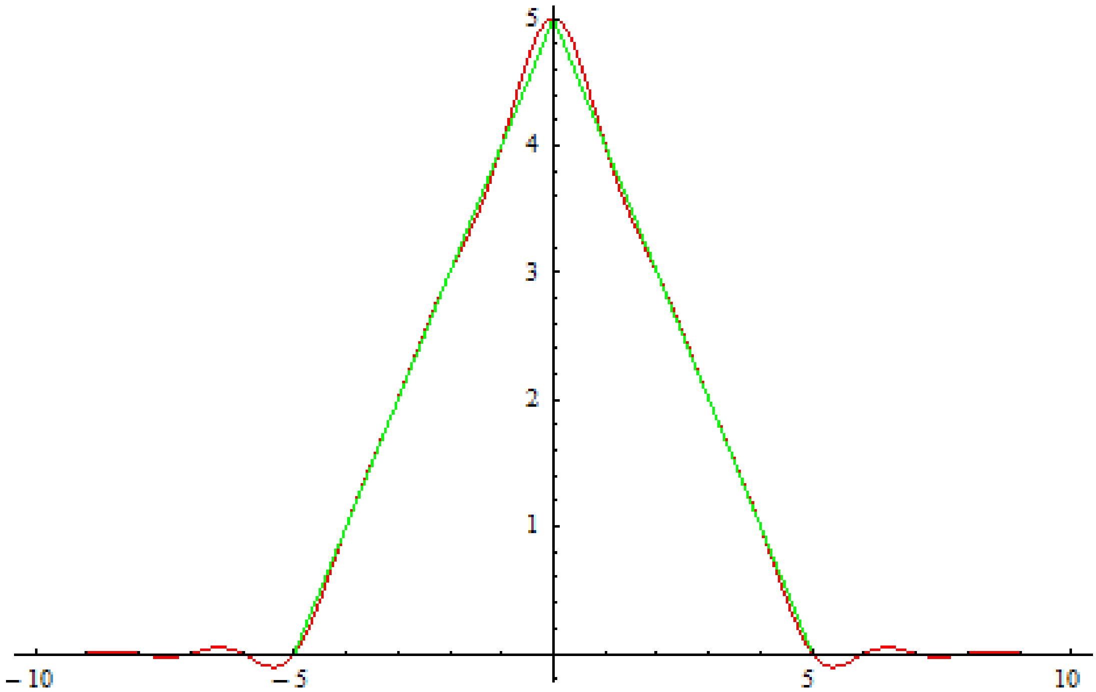

Figure 3. Interpolation of triangular signal.

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