ISSN 1066-369X, Russian Mathematics, 2018, Vol. 62, No. 4, pp. 52–59. © Allerton Press, Inc., 2018.
Original Russian Text © N.V. Malai, E.R. Shchukin, A.V. Limanskaya, 2018, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2018, No. 4, pp. 60–69.

Investigation of Boundary-Value Problem for Stationary System of Equations of Viscous Non-Isothermal Fluid

N. V. Malai[1*], E. R. Shchukin[2], and A. V. Limanskaya[1***]**

[1] *Belgorod State National Research University
ul. Pobedy 85, Belgorod, 308015 Russia*

[2] *Joint Institute for High Temperatures of the Russian Academy of Sciences
ul. Izhorskaya 13, Bld. 2, Moscow, 125412 Russia*

Received February 8, 2017

Abstract—In the Stokes approximation at small Reynolds and Peclet numbers, we obtain a solution to the boundary-value problem of flow around of particles of spherical shape for stationary system of equations of a viscous non-isothermal fluid comprising a linearized by speed Navier—Stokes equation system and the equation of heat transfer given an exponential-power law of dependence of viscosity of fluid on temperature.

DOI: 10.3103/S1066369X18040060

Keywords: *equations of hydrodynamics, the linearized by velocity Navier—Stokes equation.*

Introduction. Many phenomena observed in the atmosphere and the ocean, the problems of hydrodynamics, hydraulics, acoustics, circulatory physiology, organization of technological processes etc. are related to the motion and flow of viscous fluid around bodies, i.e., they can be studied within the framework of incompressible fluid models. In analysis of such models we use the term "relative temperature difference". By relative temperature difference we mean the ratio of the difference between the average surface temperature of the particle T_S and the temperature of the region far from it T_∞ to the latter. The relative temperature drop is considered small if the inequality $(T_S - T_\infty)/T_\infty \ll 1$ holds. When this inequality is met, the fluid transfer coefficients (viscosity, thermal conductivity, etc.) can be considered constant, and the fluid itself is called isothermal. In this case, a number of authors investigated the stationary system of hydrodynamics equations [1, 2].

If $(T_S - T_\infty)/T_\infty \sim O(1)$, then the relative temperature drop is significant. In this case, the particle is considered heated and when solving the equations of hydrodynamics and heat and mass transfer it is necessary to take into account the dependence of the molecular transfer coefficients on temperature. It complicates the analysis of the system of hydrodynamics equations, and the fluid itself is called non-isothermal. In the literature, this issue has not been adequately studied (e.g., [3—6]). The heating of a particle surface can occur due to the internal heat sources inhomogeneously distributed in its volume with the density $q_i(r, \theta)$, where r and θ are the spherical coordinates ($0 \le \theta \le \pi$). The effect of these sources can be due, for example, to the occurrence of a volumetric chemical reaction, the process of radioactive decay of a particle's substance, the absorption of electromagnetic radiation or the like. The resulting increase in the temperature of the surface of the particle affects the thermophysical characteristics of the fluid and, therefore, can significantly affect the distribution of the velocity and pressure fields in its vicinity. As shown by the non-isothermal liquid investigations, under a certain type of seek of a solution for the mass-velocity components and permissible simplifications from the viewpoint of physics, the solution of the linearized velocity system of the Navier—Stokes equations can be reduced to the solution of an ordinary inhomogeneous third-order differential equation with an isolated singular point that can be found applying functional series of a special kind.

[*] E-mail: malay@bsu.edu.ru.
[**] E-mail: evgrom@yandex.ru.
[***] E-mail: limanskayaanna@mail.ru.

1. Statement of the problem. Basic equations and boundary conditions. We consider the classical problem of the axisymmetric flow past a heated hydrosol solid spherical shape particle of radius R (inside which act unevenly distributed heat sources of power q_i) with a viscous incompressible non-isothermal fluid of velocity $\mathbf{U}_\infty \| Oz$.

The general system of hydrodynamics equations is nonlinear (due to dependence of the liquid viscosity on the temperature) and, when solving it, we make the following physical assumptions realized in most applications.

Assumption 1. Particle flow occurs at significant relative temperature differences. Among all the parameters of fluid transport, only the viscosity coefficient most strongly depends on temperature ([7], Chap. 8, P. 279). It is known [7] that the viscosity coefficient of a fluid decreases with temperature by an exponential law. Papers [3, 4] extensively applied the Reynolds formula ([7], Chap. 8, P. 299) with an error up to 40%.

In this paper we apply formula (1). Analysis of the semi-empirical formulas and experimental data available in the literature show that in order to take into account the dependence of the viscosity coefficient on temperature, the formula

$$\mu_e = \mu_\infty \left[1 + \sum_{n=1}^{N} F_n \left(\frac{T_e}{T_\infty} - 1\right)^n\right] \exp\left\{ - A\left(\frac{T_e}{T_\infty} - 1\right)\right\}, \tag{1}$$

where $\mu_\infty = \mu_e(T_\infty)$, A and F_n represent the known set of constants (its own for any specific fluid), allows to describe the fluid viscosity change in a wide range of temperatures with any necessary accuracy. For example, for water with a relative error not exceeding 2.5% in the temperature range from $273^{circ}K$ to $363^\circ K$, and for $T_\infty = 273^\circ K$ the coefficients $A = 5.779$, $F_1 = -2.318$, $F_2 = 9.118$, $F_3 = 0.00003$, $F_4 = 0.000002$, and so on. Thus, if we restrict ourselves to a relative error not exceeding 2.5%, which is often suffices for experimental calculations, then for water we can restrict ourselves to only the first two terms in the sum of (1): $F_1 = -2.318$ and $F_2 = 9.118$ ([5, 6]). Direct calculations have shown that similar situation holds for other Newtonian fluids. Here and in what follows the indices "e" and "i" refer to the viscous fluid and the heated particle, respectively, the index "∞" describes the fluid parameters at infinity in the unperturbed flow, and index "s" shows values of physical quantities taken at an average surface temperature.

Assumption 2. The thermal conductivity coefficient of a particle is much larger than that of a fluid, which is the case for most real fluids. This assumption allows us to ignore the viscosity coefficient dependence on the angle θ in the system "the particle—the fluid medium" (we assume weak angular temperature distribution asymmetry) and, consequently, the viscosity is related only to the temperature $t_{e0}(r)$, i.e., $\mu_e(t_e(r,\theta)) \approx \mu_e(t_{e0}(r))$. Then the values $t_e(r,\theta) = t_{e0}(r) + \delta t_e(r,\theta)$, here $\delta t_e(r,\theta) \ll t_{e0}(r)$; $\delta t_e(r,\theta), t_{e0}(r)$ are determined from the thermal problem solution. Under this assumption we can consider the hydrodynamic part separately from the thermal part, and the connection between them is carried out only by the boundary conditions.

Assumption 3. We consider a power-law form of a solid particle thermal conductivity dependence on the temperature: $\lambda_i = \lambda_{i0}\big(T_i/T_\infty\big)^\eta$, $-1 \le \eta \le 1$ ([8], Chap. 15, P. 339).

Assumption 4. The particle is formed by a homogeneous and isotropic in its properties substance.

Under the given assumptions we describe the flow in a spherical coordinate system related to the particle's center of mass. Here we give an axisymmetric solution to the boundary-value problem for a stationary velocity linearized equation system describing a vector field of the mass velocity $\mathbf{U}_e(x) = (U_1(x), U_2(x), U_3(x))$, the pressure $P_e(x)$ and the temperature $T_e(x)$ (Eqs. (2), (3) ([8], Chap. 2, P. 42)) in the external domain $x \in \Omega_e = \mathbb{R}^3 \setminus \Omega_i$, here Ω_i is an inner spherical region centered at zero, and also a temperature field inside the particle $T_i(x)$, $x \in \Omega_i$ (Eq. (4)):

$$\nabla P_e = \mu_e \nabla^2 \mathbf{U_e} + 2(\nabla\mu_e)\cdot\nabla\mathbf{U}_e + (\nabla\mu_e)\times(\nabla\times\mathbf{U_e}), \;\; \mathrm{div}\mathbf{U}_e = 0, \tag{2}$$

$$\Delta T_e = 0, \tag{3}$$

$$\operatorname{div}(\lambda_i \nabla T_i) = -q_i, \tag{4}$$

here $\nabla = (\nabla_1, \nabla_2, \nabla_3)$ is a vector differential Hamilton operator in Cartesian coordinates, $\nabla_j \equiv \partial/\partial x_j$, q_i is a function defined in Ω_i, which describes the thermal sources density within a particle.

System (2)–(4) is solved with the following boundary conditions in the spherical coordinate system $(y = r/R, \varphi, \theta)$, by taking into account adhesion condition (5) on the particle surface $(y = 1)$ for normal $(U_r\,(y, \theta))$ and tangent $(U_\theta\,(y, \theta))$ components of the mass velocity $\mathbf{U}_e$, the equality of temperatures and the continuity of the radial heat fluxes given by (6), and standard conditions (7) for $y \to \infty$ (far from the particle) ([8], Chap. 4.18, P. 145) and (8) for $y \to 0$ (inside the particle) ($U_\infty = |\mathbf{U}_\infty|$):

$$\lim_{y\to 1} U_r\,(y, \theta) = 0, \;\; \lim_{y\to 1} U_\theta\,(y, \theta) = 0, \tag{5}$$

$$\lim_{y\to 1} T_e\,(y, \theta) = \lim_{y\to 1} T_i\,(y, \theta)\,, \;\; \lim_{y\to 1} \left(\lambda_e \frac{\partial T_e\,(y, \theta)}{\partial y}\right) = \lim_{y\to 1} \left(\lambda_i \frac{\partial T_i\,(y, \theta)}{\partial y}\right), \tag{6}$$

$$\lim_{y\to\infty} U_r\,(y, \theta) = U_\infty \cos\theta, \;\; \lim_{y\to\infty} U_\theta\,(y, \theta) = -U_\infty \sin\theta, \;\; \lim_{y\to\infty} P_e = P_\infty, \;\; \lim_{y\to\infty} T_e = T_\infty, \tag{7}$$

$$\lim_{y\to 0} |T_i| < \infty. \tag{8}$$

The boundary conditions for the mass-velocity components far from the particle (7) form tells us that the solution for the mass-velocity component $U_r(y, \theta)$, $U_\theta(y, \theta)$ and the pressure $P_e(y, \theta)$ can be found in the form of Legendre and Gegenbauer polynomial expansions. We need them in order to find the total force acting on the particle ([9], Chap. II, P. 69) by integrating the stress tensor over the particle surface

$$\lim_{r\to R} F_z(r) = \lim_{r\to R} \int_S \left(-P_e(r/R, \theta)\cos\theta + \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right) r^2 \sin\theta\, d\theta\, d\varphi. \tag{9}$$

It can be shown that this force is determined only by the first terms of the expansions ([8], Chap. 4, P. 156). Therefore, we assume that

$$U_r\,(y, \theta) = U_\infty G\,(y) \cos\theta, \quad U_\theta\,(y, \theta) = -U_\infty g\,(y) \sin\theta, \tag{10}$$

here $G\,(y)$ and $g\,(y)$ are the unknown functions depending on the radial coordinate.

2. Solution to the heat transfer equations and the Navier–Stokes equation linearized in velocity. In a spherical coordinate system the system of equations for a viscous incompressible nonisothermal fluid describing the the velocity and the pressure distribution outside the particle has the following form ([9], Chap. II, P. 70):

$$\frac{\partial P_e}{\partial y} = \frac{\partial \sigma_{rr}}{\partial y} + \frac{2}{y}\sigma_{rr} + \frac{1}{y}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\cot\theta}{y}\sigma_{r\theta} - \frac{\sigma_{\theta\theta} - \sigma_{\varphi\varphi}}{y}, \tag{11}$$

$$\frac{\partial P_e}{\partial \theta} = y\frac{\partial \sigma_{r\theta}}{\partial y} + 3\sigma_{r\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \cot\theta\left(\sigma_{\theta\theta} - \sigma_{\varphi\varphi}\right), \tag{12}$$

$$\frac{1}{y^2}\frac{\partial}{\partial y}\left(y^2 U_r\right) + \frac{1}{y\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, U_\theta\right) = 0, \tag{13}$$

here σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ and $\sigma_{\varphi\varphi}$ are the components of the stress tensor in a spherical coordinate system, defined by the equalities ([9], Chap. II, P. 70)

$$\sigma_{rr} = 2\mu\frac{\partial U_r}{\partial y}, \;\; \sigma_{\theta\theta} = \mu\left(\frac{2}{y}\frac{\partial U_\theta}{\partial \theta} + \frac{2}{y}U_r\right),$$

$$\sigma_{\varphi\varphi} = \mu\left(\frac{2}{y}U_r + \frac{2}{y}\cot\theta\, U_\theta\right), \;\; \sigma_{r\theta} = \mu\left(\frac{\partial U_\theta}{\partial y} + \frac{1}{y}\frac{\partial U_r}{\partial \theta} - \frac{U_\theta}{y}\right).$$

Let us find the temperature fields outside and inside the particle. Separation of variables allows us to solve Eqs. (3), (4) and obtain the following solutions satisfying boundary conditions (6)–(8):

$$t_{e0}(y) = 1 + \gamma/y, \tag{14}$$

$$t_{i0}(y) = \left(B + \frac{1+\eta}{4\pi R\lambda_{i0}T_\infty y}\int_V q_i dV + \int_1^y \frac{\psi_0}{y}dy - \frac{1}{y}\int_1^y \psi_0 dy\right)^{\frac{1}{1+\eta}},$$

here $\psi_0 = -\frac{R^2(1+\eta)}{2\lambda_{i0}T_\infty}y^2\int\limits_{-1}^{+1} q_i d(\cos\theta)$, $\gamma = t_S - 1$ is a dimensionless parameter characterizing the heating of the particle surface, $t_S = T_S/T_\infty$, T_S is the average surface temperature of the particle defined by equality (15), and the integration is performed over the entire volume of the particle:

$$\frac{T_S}{T_\infty} = 1 + \frac{1}{4\pi R\lambda_e T_\infty}\int_V q_i dV. \tag{15}$$

From (15) we see that the density of thermal sources q_i has a significant influence on the value of the average surface temperature of a particle.

By formula (14) and Assumption 2 expression (1) turns into

$$\mu_e = \mu_\infty\left(1 + \sum_{n=1}^{N} F_n\frac{\gamma^n}{y^n}\right)\exp\left(-\frac{A\gamma}{y}\right). \tag{16}$$

We find the connection between the functions $G\,(y)$ and $g\,(y)$ from continuity equation (13) taking into account expressions (10):

$$g\,(y) = G\,(y) + \frac{y}{2}\frac{dG\,(y)}{dy}. \tag{17}$$

Substituting expressions (16), (17) into (12) and acting as in [10] we eventually obtain the following nonhomogeneous third-order differential equation for the function $G\,(y)$ on the interval $y \in [1,\infty)$ (for the sake of simplicity we assume that $N = \infty$ due to the explanations of formula (1)):

$$y^4\sum_{n=0}^{\infty}\alpha_n^{(1)}\frac{\gamma^n}{y^n}\frac{d^3G(y)}{dy^3} + y^3\sum_{n=0}^{\infty}\alpha_n^{(2)}\frac{\gamma^n}{y^n}\frac{d^2G(y)}{dy^2} + y^2\sum_{n=0}^{\infty}\alpha_n^{(3)}\frac{\gamma^n}{y^n}\frac{dG(y)}{dy} = D\sum_{n=0}^{\infty}\alpha_n^{(4)}\frac{\gamma^n}{y^n}, \tag{18}$$

here for $n \geq 1$ $\alpha_n^{(1)} = F_n$, $\alpha_n^{(2)} = (4-n)F_n + AF_{n-1}$, $\alpha_n^{(3)} = -2(n+2)F_n + 2AF_{n-1}$, $\alpha_n^{(4)} = A^n/n!$, D is a constant to be determined from boundary conditions (19), $\alpha_0^{(1)} = \alpha_0^{(4)} = F_0 = 1$, $\alpha_0^{(2)} = 4$, $\alpha_0^{(3)} = -4$.

Then boundary conditions (5), (7) turn into

$$\lim_{y\to 1} G(y) = 0,\ \lim_{y\to 1}\left(G(y) + \frac{y}{2}\frac{dG}{dy}\right) = 0,\ \lim_{y\to\infty} G(y) = 1. \tag{19}$$

Note that one of the homogeneous equation corresponding to non-homogeneous equation (18) solutions is the function $G(y) = \text{const}$.

Let us find the solution to Eq. (18) in the interval $[1,\infty)$ and prove the uniqueness of solution to boundary-value problem (18)–(19). To do this, we consider the new variable $\xi = 1/y$ in Eq. (18), introduce a new function $\Phi = dG/dy$ and finally find the following equation:

$$\xi^2\sum_{n=0}^{\infty}\alpha_n^{(1)}\gamma^n\xi^n\frac{d^2\Phi(\xi)}{d\xi^2} + \xi\sum_{n=0}^{\infty}\left(2\alpha_n^{(1)} - \alpha_n^{(2)}\right)\gamma^n\xi^n\frac{d\Phi(\xi)}{d\xi} + \sum_{n=0}^{\infty}\gamma^n\xi^n\alpha_n^{(3)}\frac{d\Phi(\xi)}{d\xi} = \xi^2 D\sum_{n=0}^{\infty}\alpha_n^{(4)}\gamma^n\xi^n\Phi(\xi). \tag{20}$$

The point $\xi = 0$ is a regular singular point for a homogeneous equation corresponding to the non-homogeneous equation (20) [11, 12], therefore the solution to the homogeneous differential equation can be sought with the help of generalized power series.

$$\Phi(\xi) = \xi^\rho \sum_{n=0}^{\infty} \Delta_n \xi^n \ \ (\Delta_0 \neq 0). \tag{21}$$

Calculating the derivatives we obtain

$$\Phi'(y) = \sum_{n=0}^{\infty}(n+\rho)\Delta_n y^{n+\rho-1}, \ \ \Phi''(y) = \sum_{n=0}^{\infty}(n+\rho)(n+\rho-1)\Delta_n y^{n+\rho-2}. \tag{22}$$

Put (21), (22) into a homogeneous equation corresponding to inhomogeneous equation (20), and equate the coefficients of $\xi^{n+\rho}$ in order to obtain the defining equation

$$\rho^2 - 3\rho - 4 = 0, \tag{23}$$

the roots of which are $\rho_1 = 4$, $\rho_2 = -1$. Note that the root difference is integer, therefore, according to the general theory of differential equation solutions in the form of generalized power series by the Frobenius method, all other solutions, in addition to the first one corresponding to $\rho_1 = 4$, contain an additional summand with the factor $\ln y$ multiplied by the first solution [11, 12].

Thus, by taking into account the root values of the defining equation we obtain a system of linearly independent solutions for the homogeneous equation corresponding to non-homogeneous equation (20) in the form

$$\Phi_1(\xi) = \xi^4 \sum_{n=0}^{\infty} \Delta_n^{(1)} \xi^n, \ \ \Phi_3(\xi) = \frac{1}{\xi}\sum_{n=0}^{\infty} \Delta_n^{(3)} \xi^n + \beta \ln\xi\, \Phi_1(\xi). \tag{24}$$

A partial solution to inhomogeneous equation (20) is the function

$$\Phi_2(\xi) = D\Phi_4(\xi), \ \ \Phi_4(\xi) = \xi^2 \sum_{n=0}^{\infty} \Delta_n^{(2)} \xi^n + \alpha \ln\xi\, \Phi_1(\xi). \tag{25}$$

Put (24), (25) into Eq. (20) and obtain the recurrent formulas for the coefficients

$$\Delta_n^{(1)} = -\frac{1}{n(n+5)} \sum_{k=1}^{n} \left\{ (n+4-k)\left[\alpha_k^{(1)}(n+5-k) - \alpha_k^{(2)}\right] + \alpha_k^{(3)} \right\} \gamma^k \Delta_{n-k}^{(1)} \ \ (n \geq 1),$$

$$\Delta_n^{(2)} = -\frac{1}{(n+3)(n-2)} \left\{ -6\alpha_k^{(4)}\gamma^n + \sum_{k=1}^{n}\left[(n-k+2)[\alpha_k^{(1)}(n-k+3) - \alpha_k^{(2)}] + \alpha_k^{(3)}\right]\gamma^k \Delta_{n-k}^{(2)} \right.$$
$$\left. + \alpha \sum_{k=0}^{n}\left[\alpha_k^{(1)}(2n-2k+5) - \alpha_k^{(2)}\right]\gamma^k \Delta_{n-k-2}^{(1)} \right\} \ \ (n \geq 3),$$

$$\Delta_n^{(3)} = -\frac{1}{n(n-5)} \left\{ \sum_{k=1}^{n}\left[(n-k-1)[\alpha_k^{(1)}(n-k) - \alpha_k^{(2)}] + \alpha_k^{(3)}\right]\gamma^k \Delta_{n-k}^{(3)} \right.$$
$$\left. + \beta \sum_{k=0}^{n-5}\left[\alpha_k^{(1)}(2n-2k-1) - \alpha_k^{(2)}\right]\gamma^k \Delta_{n-k-5}^{(1)} \right\} \ \ (n \geq 6).$$

When calculating the coefficients $\Delta_n^{(i)}$, $i = 1, 2, 3$, by the above formulas it is necessary to note that $\Delta_0^{(1)} = -3$, $\Delta_0^{(2)} = -1$, $\Delta_2^{(2)} = 1$, $\Delta_1^{(2)} = -\gamma[6\alpha_1^{(4)} + 2(3\alpha_1^{(1)} - \alpha_1^{(2)}) + \alpha_1^{(3)}]/4$, $\alpha = \frac{\gamma}{15}\{-6\alpha_2^{(4)}\gamma + [3(4\alpha_1^{(1)} - \alpha_1^{(2)}) + \alpha_1^{(3)}]\Delta_1^{(2)} - [2(3\alpha_2^{(1)} - \alpha_2^{(2)}) + \alpha_2^{(3)}]\gamma\}$, $\Delta_0^{(3)} = \gamma(\alpha_1^{(3)} + \alpha_1^{(2)})/4$, $\Delta_1^{(3)} = [\alpha_1^{(3)}\gamma\Delta_1^{(3)} +$

$(\alpha_2^{(3)}+\alpha_2^{(2)})\gamma^2\big]/6$, $\Delta_3^{(3)}=\big[(2\alpha_1^{(1)}-\alpha_1^{(2)}+\alpha_1^{(3)})\gamma\Delta_2^{(3)}+\alpha_2^{(3)}\gamma^2\Delta_1^{(3)}+(\alpha_3^{(3)}+\alpha_3^{(2)})\gamma^3\big]/6$, $\Delta_4^{(3)}=\big[(6\alpha_1^{(1)}-2\alpha_1^{(2)}+2\alpha_1^{(3)})\gamma\Delta_3^{(3)}+(2\alpha_2^{(1)}-\alpha_2^{(2)}+\alpha_2^{(3)})\gamma^2\Delta_2^{(3)}+\alpha_3^{(4)}\gamma^3\Delta_1^{(3)}+(\alpha_4^{(3)}+\alpha_4^{(2)})\gamma^4\big]/4$, $\Delta_5^{(3)}=1$, $\beta=-\big[(12\alpha_1^{(1)}-3\alpha_1^{(2)}+3\alpha_1^{(3)})\gamma\Delta_4^{(3)}+(6\alpha_2^{(1)}-2\alpha_2^{(2)}+2\alpha_2^{(3)})\gamma^2\Delta_3^{(3)}+(2\alpha_3^{(1)}-\alpha_3^{(2)}+\alpha_3^{(3)})\gamma^3\Delta_2^{(3)}+\alpha_4^{(3)}\gamma^4\Delta_1^{(3)}+(\alpha_5^{(3)}+\alpha_5^{(2)})\gamma^5\big]/4$.

Thus, the general solution to Eq. (20) has the form

$$\Phi(\xi)=C_1\Phi_1(\xi)+\Phi_2(\xi)+C_2\Phi_3(\xi).$$

The function $\Phi(\xi)=C_1\Phi_1(\xi)+\Phi_2(\xi)+C_2\Phi_3(\xi)$ meets Eq. (20) by construction. The series of the functions $\Phi_i(\xi)$, $i=1,2,3$, uniformly converge for $\xi\in(0,1)$.

Returning again to the variable y, we obtain the general solution to Eq. (18)

$$G(y)=A_0+A_1G_1(y)+A_2G_2(y)+A_3G_3(y), \tag{26}$$

here

$$G_1(y)=-\frac{1}{y^3}\sum_{n=0}^{\infty}\frac{\Delta_n^{(1)}}{(n+3)y^n},\ \ G_2(y)=-\frac{1}{y}\sum_{n=0}^{\infty}\frac{\Delta_n^{(2)}}{(n+1)y^n}-\frac{\alpha}{y^3}\sum_{n=0}^{\infty}\left[(n+3)\ln\frac{1}{y}-1\right]\frac{\Delta_n^{(1)}}{(n+3)^2y^n},$$
$$G_3(y)=\frac{y^2}{2}+y\Delta_1^{(3)}+\Delta_2^{(3)}\ln y-y^2\sum_{n=3}^{\infty}\frac{\Delta_n^{(3)}}{(n-2)y^n}-\frac{\beta}{y^3}\sum_{n=0}^{\infty}\left[(n+3)\ln\frac{1}{y}-1\right]\frac{\Delta_n^{(1)}}{(n+3)^2y^n}. \tag{27}$$

The choice of constants $\Delta_0^{(1)}$, $\Delta_0^{(2)}$ is performed so that the functions $G_1(y)$ and $G_2(y)$ tend to the corresponding functions ([8], Chap. 4, P. 140) for the sphere at small relative temperature drops. Note that the solution $G_3(y)$ to Eq. (18) does not meet boundary condition (19) for $y\to\infty$.

The constants A_0, A_1, A_2, and A_3 are uniquely determined from boundary conditions (19). Clearly, $A_0=1$, $A_3=0$, and we have the following equation system for the constants A_1, A_2:

$$\begin{gathered}1+A_1G_1(1)+A_2G_2(1)=0,\\ 1+A_1[G_1(1)+\frac{1}{2}G_1'(1)]+A_2\big[G_2(1)+\frac{1}{2}G_2'(1)\big]=0.\end{gathered} \tag{28}$$

System (28) has a unique solution due to the linear independence of the solutions $G_1(y)$, $G_2(y)$.

As a result of this study, we proved

Theorem. *Function $G(y)=A_0+A_1G_1(y)+A_2G_2(y)$ with the coefficients A_1, A_2 defined by system* (28)*, and $A_0=1$, is the only solution of Eq.* (18) *subject to boundary conditions* (19).

We now determine the mass-velocity components $\mathbf{U}_e$ and the pressure P_e, which allow to reconstruct the total force acting on a nonuniformly heated particle moving in a nonisothermal fluid. Taking into account (10), (17) and the proved theorem, we have

$$U_r(y,\theta)=U_\infty\Big[1+A_1G_1(y)+A_2G_2(y)\Big]\cos\theta, \tag{29}$$

$$U_\theta(y,\theta)=-U_\infty\left[1+A_1\big(G_1(y)+\frac{y}{2}G_1'(y)\big)+A_2\big(G_2(y)+\frac{y}{2}G_2'(y)\big)\right]\sin\theta. \tag{30}$$

Since we know the explicit form of the functions $U_r(y,\theta)$ and $U_\theta(y,\theta)$ from (29), (30), relation (12) makes it possible to find the pressure field expression

$$P_e(y,\theta)=P_\infty+\frac{\mu_eU_\infty}{R}\left[\frac{y^2}{2}\frac{d^3G(y)}{dy^3}+3y\frac{d^2G(y)}{dy^2}+2\frac{dG(y)}{dy}+\frac{1}{\mu_e}\frac{d\mu_e}{dy}\left(\frac{y^2}{2}\frac{d^2G(y)}{dy^2}+y\frac{dG(y)}{dy}\right)\right]\cos\theta.$$

The total force acting on the particle is determined by integrating the stress tensor over its surface in a spherical coordinate system applying formula (9). Integrating (9) over the angles ($0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$), we obtain

$$\mathbf{F}_z = 6\pi R\mu_{e\infty}U_\infty f_\mu \mathbf{n}_z, \;\; f_\mu = \frac{2N_2}{3N_1}, \tag{31}$$

here $\mathbf{n}_z$ is the a unit vector on the axis Oz, $N_1 = G_1(1)G_2'(1) - G_2(1)G_1'(1)$, $N_2 = -G_1'(1)$.

A spherical particle falling under the action of gravity in a viscous non-isothermal fluid acquires a constant velocity $\mathbf{U}_p$ as soon as the gravity is balanced by the hydrodynamic forces.

The gravity force acting on a particle, balanced by the buoyancy force, equals

$$\mathbf{F}_g = (\rho_{iS} - \rho_{eS})g\frac{4}{3}\pi R^3\mathbf{n}_z, \tag{32}$$

where g is the gravity acceleration, ρ_{iS} and ρ_{eS} are the particle and fluid density taken at an average surface temperature of the particle equal T_S.

Equating (31) to (32) and taking into account that $\mathbf{U}_\infty = -\mathbf{U}_p$ we obtain an expression for the steady-state drop rate of a solid nonuniformly heated particle of spherical shape in the gravity force field and in a viscous non-isothermal fluid

$$\mathbf{U}_p = h_\mu \mathbf{n}_z, \;\; h_\mu = \frac{2}{9}\frac{\rho_{iS} - \rho_{eS}}{\mu_\infty f}R^2 g. \tag{33}$$

So, formulas (31) and (33) make it possible to estimate the force acting on a nonuniformly heated sphere and the rate of its gravitational incidence by taking into account the exponential-power form of the fluid viscosity coefficients dependence on temperature for arbitrary relative temperature differences between the particle surface and an domain far from it.

When the heating of the particle surface is sufficiently small, i.e., the average temperature of the particle surface is insignificantly different from the ambient temperature far from it ($\gamma = 0$), the dependence of the viscosity coefficient on temperature can be neglected and formulas (31) and (33) turn into the known expressions for the sphere, obtained by Stokes for small relative temperature drops ([8], P. 146):

$$\mathbf{F}_S = 6\pi R\mu_{e\infty}U_\infty \mathbf{n}_z, \;\; \mathbf{U}_p = \frac{2}{9}\frac{\rho_{i\infty} - \rho_{e\infty}}{\mu_{e\infty}}R^2 g\mathbf{n}_z.$$

For large relative differences the situation is completely different. The constant $\gamma = T_S/T_\infty - 1$ depends on the average surface temperature of the particle T_S. In the case of uneven heating of the surface, T_S is determined from Eq. (15) and depends on the density of thermal sources that are inhomogeneously distributed in the bulk of the particle. It follows that both the functions G_1, G_2 and the viscosity coefficient also depend on the thermal sources density, because these functions include the value γ, ultimately the same holds for the resistance force magnitude and the rate of incidence. In fact, we can regulate the behavior of a non-uniformly heated solid particle in a viscous non-isothermal fluid.

In order to illustrate the comparison of the viscosity and the temperature contribution dependence in the particle incidence rate in water at $T_\infty = 273\ ^{circ}K$, we give the curves that relate the values of the ratio $h = h_\mu / h_\mu|_{T_S = T_\infty}$ with the average surface temperature T_S of large granite particles with a radius of $R = 10^{-5}$ m (see Figure).

The expression for the function h^* is taken for small relative temperature drops ([8], P. 146), but the viscosity is taken into account directly at the average surface temperature of the particle, i.e.,

$$\mathbf{F}_S = 6\pi R\mu_{e\infty}U_\infty \mathbf{n}_z, \;\; \mathbf{U}_p = h_\mu^* \mathbf{n}_z, \;\; h_\mu^* = \frac{2}{9}\frac{\rho_{i\infty} - \rho_{e\infty}}{\mu_e^{(s)}}R^2 g.$$

One can see from the graphs that the formulas give a significant error for small temperature differences, even if the viscosity is taken into account directly at the average surface temperature of the particle.

The numerical analysis carried out using the given formulas shows the nonlinear dependence of the gravitational motion force and velocity on the average surface temperature of the particle. Analogous

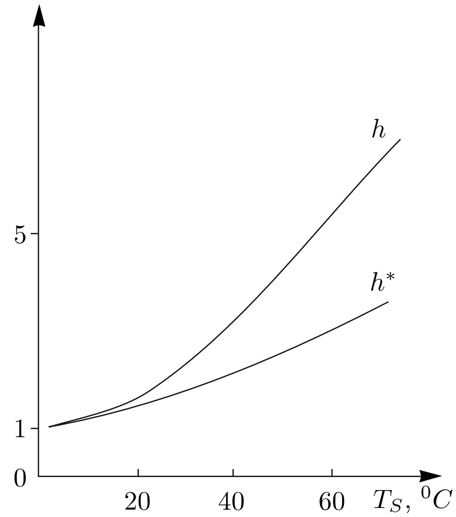

Figure.

results also hold in the case of a heated spherical shape solid particle in gaseous medium motion [13, 14]. The results obtained for a system of hydrodynamic equations linearized in velocity allow to describe a wide class of other physical problems, for example, deposition of particles in different-temperature channels, sounding of the atmosphere by high-power laser radiation, development of methods for fine cleaning fluids from hydrosol impurities or the like.

REFERENCES

1. Ladyzhenskaya, O. A. "Untersuchung der Navier–Stokesschen Gleichung für den Fall der stationären Bewegung einer inkompressiblen Flüssigkeit", Usp. Mat. Nauk **XIV** No. 3(87), 75–97 (1959) [in Russian].
2. Ladyzhenskaya, O. A. "The Sixth Millennium Problem: Navier–Stokes Equations, Existence and Smoothness", Russ. Math. Surv. **58**, No. 2, 251–286 (2003).
3. Naidenov, V. I. "Steady Flow of a Viscous Incompressible Fluid Taking Temperature Dependence of Viscosity Into Account", J. Appl. Math. Mech. **38**, No. 1, 144–148 (1974).
4. Golovin, A. M., Fominykh, V. V. "Motion of a Spherical Particle in a Viscous Nonisothermal Fluid", Fluid Dyn. **18**, 26–29 (1983).
5. Malay, N. V., Pleskanev, A. A., Shchukin, E. R. "Effect of Internal Heat Evolution on the Motion of a Solid Particle in a Viscous Fluid", Technical Physics **76**, No. 3, 317–321 (2006).
6. Malaj, N. V., Shchukin, E. R., Yalamov, Yu. I. "Thermocapillary Drift of a Heated Droplet in a Viscous Liquid in the Field of Electromagnetic Radiation", Doklady Physics **379**, No. 6, 1–6 (2001).
7. Bretschneider, S. *Properties of Gases and Liquids. Engineering Methods of Calculation* (Khimiya, Moscow, 1966) [Russian translation].
8. Happel, J., Brenner, H. *Low Reynolds Number Hydrodynamics* (Noordhoff, Leyden, 1965).
9. Landau, L. D., Lifshits, E. M. *Fluid Mechanics* (ITTL, Moscow, 1954; Pergamon Press, Oxford–Elmsford, New York, 1987), Vol. 6.
10. Glushak, A. V., Malaj, N. V., Mironova, N. N. "Solution of the Boundary Value Problem for the Navier–Stokes Equation for the Flow of a Gaseous Medium Past a Heated Spheroid", Differ. Equ. **48** (6), 886–890 (2012).
11. Kamke, E. *Differentialgleichungen. Lösungsmethoden und Lösungen I. Gewöhnliche Differentialgleichungen* (Leipzig, 1942; GIFML, Moscow, 1981).
12. Coddington, E. A., Levinson, N. *Theory of Ordinary Differential Equations* (New York, McGraw-Hill, 1955; In. Lit., Moscow, 1958).
13. Malai, N. V., Shchukin, E. R., Stukalov, A. A., Ryazanov, K. S. "Gravity-Induced Motion of a Uniformly Heated Solid Particle in a Gaseous Medium", Journal of Applied Mechanics and Technical Physics **49**, No. 1, 58–63 (2008).
14. Malai, N. V., Shchukin, E. R., Shulimanova, Z. L. "Molecular Heat Exchange of a Solid Spherical Particle With a Gaseous Medium", High Temperature **51**, No. 4, 495–499 (2013).

Translated by P. N. Ivan'shin