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MATHEMATICAL MODELS OF SEISMICS IN COMPOSITE MEDIA: ELASTIC AND PORO-ELASTIC COMPONENTS

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Abstract. In the present paper we consider elastic and poroelastic media having a common interface. We derive the macroscopic mathematical models for seismic wave propagation through these two different media as a homogenization of the exact mathematical model at the microscopic level. They consist of seismic equations for each component and boundary conditions at the common interface, which separates different media. To do this we use the two-scale expansion method in the corresponding integral identities, defining the weak solution. We illustrate our results with the numerical implementations of the inverse problem for the simplest model.

1. Introduction

This article is devoted to a description of seismic wave propagation in composite media $Q \subset \mathbb{R}^3$, consisting of the elastic medium $\Omega^{(0)}$, poroelastic medium Ω, which is perforated by a periodic system of pores filled with a fluid, and common interface $S^{(0)}$ between $\Omega^{(0)}$ and Ω (see Figures 1, 2). That is, $Q = \Omega \cup S^{(0)} \cup \Omega^{(0)}$ and $\Omega = \Omega_f \cup \Gamma \cup \Omega_s$, where Ω_s is a solid skeleton, Ω_f is a pore space (liquid domain), and Γ is a common boundary "solid skeleton-liquid domain".

The structure of the heterogeneous medium Q is too complicated and makes hard a numerical simulation of seismic waves propagation in multiscale media. The main difficulty here is a presence of both components (solid and liquid) in each sufficiently small subdomain of Q. It requires to change the governing equations (from Lame's equations to the Stokes equations) at the scale of some tens microns.

There are two basic methods to describe physical processes in such media: the phenomenological method and the asymptotical one which is based on the upscaling approaches. The phenomenological approach for waves propagation through a poroelastic medium [4, 5] leads, in particular, to Biot model [1]-[3]. It based on the system of axioms (relations between the parameters of the medium), which define the given physical process. But, there can be another system of axioms defining the same process. Thus, it is necessary choose the correct authenticity criterion of the mathematical description of the process. It can be, for example, the physical experiment. As a rule, each phenomenological model contains some

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set of phenomenological constants. Therefore, one can achieve agreement between the suggested theory and selected series of experiments changing these parameters.

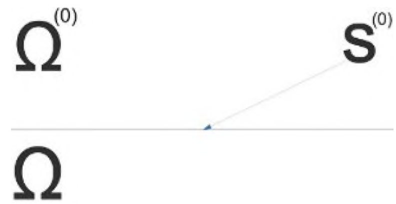

FIGURE 1. Domain in consideration

The second method, suggested by Burridge and Keller [6] and Sanchez-Palencia [7], based on the homogenization. It consists of:

(1) an exact description of the process at the microscopic level based on the fundamental laws of continuum mechanics,and

(2) the rigorous homogenization of the obtained mathematical model.

To explain the method we consider a characteristic function $\chi_0(\mathbf{x})$ of the pore space Ω_f. Let L is the characteristic size of the physical domain in consideration, τ is the characteristic time of the physical process, ρ^0 is the mean density of water, and g is acceleration due gravity. In dimensionless variables

$$\mathbf{x} \to \frac{\mathbf{x}}{L}, \quad \mathbf{w} \to \alpha_\tau \frac{\mathbf{w}}{L}, \quad t \to \frac{t}{\tau}, \quad \mathbf{F} \to \frac{\mathbf{F}}{g}, \quad \rho \to \frac{\rho}{\rho^0},$$

the dynamic system for the displacements $\mathbf{w}$ and pressure p of the medium takes the form [6, 7, 8]:

$$\varrho \frac{\partial^2 \mathbf{w}}{\partial t^2} = \nabla \cdot \mathbb{P} + \varrho \mathbf{F}, \tag{1.1}$$

$$\mathbb{P} = \chi_0\, \alpha_\mu \mathbb{D}(x, \frac{\partial \mathbf{w}}{\partial t}) + (1-\chi_0)\alpha_\lambda\, \mathbb{D}(x, \mathbf{w}) + \big(\chi_0 \alpha_\nu (\nabla \cdot \frac{\partial \mathbf{w}}{\partial t}) - p\big)\mathbb{I}, \tag{1.2}$$

$$p + \alpha_p \nabla \cdot \mathbf{w} = 0. \tag{1.3}$$

Equations (1.1)-(1.3) are understood in the sense of distributions as corresponding integral identities. They are equivalent to the Stokes equations

$$\varrho_f \frac{\partial \mathbf{v}}{\partial t} = \nabla \cdot \mathbb{P}_f + \varrho_f \mathbf{F}, \quad \frac{\partial p}{\partial t} + \alpha_{p,f} \nabla \cdot \mathbf{v} = 0, \tag{1.4}$$

$$\mathbb{P}_f = \alpha_\mu \mathbb{D}(x, \mathbf{v}) + \big(\alpha_\nu (\nabla \cdot \mathbf{v}) - p\big)\mathbb{I} \tag{1.5}$$

for the velocity $\mathbf{v} = \frac{\partial \mathbf{w}}{\partial t}$ and pressure p in the pore space Ω_f and the Lame equations

$$\varrho_s \frac{\partial^2 \mathbf{w}}{\partial t^2} = \nabla \cdot \mathbb{P}_s + \varrho_s \mathbf{F}, \quad p + \alpha_{p,s} \nabla \cdot \mathbf{w} = 0, \tag{1.6}$$

$$\mathbb{P}_s = \alpha_\lambda \mathbb{D}(x, \mathbf{w}) - p\mathbb{I} \tag{1.7}$$

for the solid displacements $\mathbf{w}$ and pressure p in Ω_s.

At the common boundary Γ velocities and normal tensions are continuous:

$$\frac{\partial \mathbf{w}}{\partial t} = \mathbf{v}, \quad \mathbb{P}_s \cdot \mathbf{n} = \mathbb{P}_f \cdot \mathbf{n}. \tag{1.8}$$

Here $\mathbf{n}$ is a unit normal to Γ.

In (1.1)-(1.8), $\mathbb{D}(x,\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^*)$ is the symmetric part of $\nabla\mathbf{u}$, $\mathbb{I}$ is a unit tensor, $\mathbf{F}$ is a given vector of distributed mass forces,

$$\alpha_p = \alpha_{p,f}\chi_0 + \alpha_{p,s}(1-\chi_0), \quad \varrho = \varrho_f\,\chi_0 + \varrho_s(1-\chi_0),$$
$$\alpha_\tau = \frac{L}{g\tau^2}, \quad \alpha_\mu = \frac{2\mu}{\alpha_\tau \tau Lg\,\rho^0}, \quad \alpha_\lambda = \frac{2\lambda}{\alpha_\tau Lg\,\rho^0},$$
$$\alpha_\nu = \frac{2\nu}{\alpha_\tau \tau Lg\,\rho^0}, \quad \alpha_{p,f} = \frac{\varrho_f c_f^2}{\alpha_\tau Lg}, \quad \alpha_{p,s} = \frac{\varrho_s c_s^2}{\alpha_\tau Lg},$$

where μ is the dynamic viscosity, ν is the bulk viscosity, λ is the elastic constant, ϱ_f and ϱ_s are the respective mean dimensionless densities of the liquid in pores and the solid skeleton, correlated with the mean density of water ρ^0, and c_f and c_s are the speed of compression sound waves in the pore liquid and in the solid skeleton respectively.

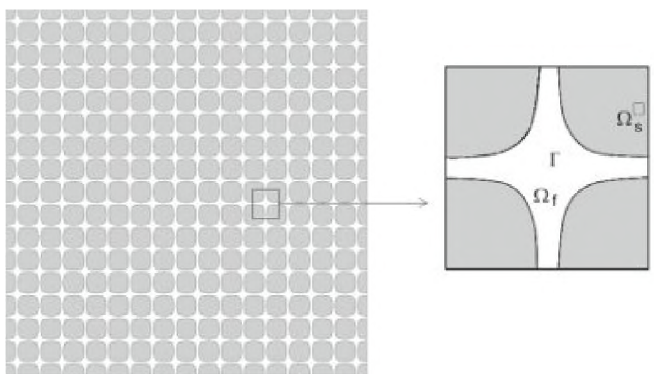

FIGURE 2. The pore structure

The mathematical model (1.1)–(1.3) can not be useful for practical needs, since the function χ_0 changes its value from 0 to 1 on the scale of a few microns. Fortunately, the system possesses a natural small parameter $\varepsilon = \frac{l}{L}$, where l is the average size of pores. Thus, the most suitable way to get a practically significant mathematical model, which approximate (1.1)–(1.3), is a homogenization or upscaling. That is, we suppose the ε-periodicity of the solid skeleton, let ε to be variable, and look for the limit in (1.1)–(1.3) as $\varepsilon \to 0$.

There are different homogenized (limiting) systems, depending on of $\alpha_\mu, \alpha_\lambda$, ... Some of these numbers might be small and some might be large. We may represent them as a power of ε, or as functions depending on ε.

Let

$$\mu_0 = \lim_{\varepsilon\searrow 0} \alpha_\mu(\varepsilon), \quad \nu_0 = \lim_{\varepsilon\searrow 0} \alpha_\nu(\varepsilon), \quad \lambda_0 = \lim_{\varepsilon\searrow 0} \alpha_\lambda(\varepsilon),$$
$$c_{f,0}^2 = \lim_{\varepsilon\searrow 0} \alpha_{p,f}(\varepsilon), \quad c_{s,0}^2 = \lim_{\varepsilon\searrow 0} \alpha_{p,s}(\varepsilon),$$
$$\mu_1 = \lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon^2}, \quad \lambda_1 = \lim_{\varepsilon\searrow 0} \frac{\alpha_\lambda}{\varepsilon^2}.$$

It is clear that the choice of these limits depend on our willing. For example, for $\varepsilon = 10^{-2}$ and $\alpha = 2\cdot 10^{-1}$ we may state that $\alpha = 2\cdot\varepsilon^{-\frac{1}{2}}$, or $\alpha = 0.02\cdot\varepsilon^0$. It is usual procedure when we neglect some terms in differential equations with small coefficients and get more simple equations, still describing the physical process.

The detailed analyses of all possible limiting regimes has been done in [8, 9]. To describe the seismic in two different media (elastic and poroelastic), having a common interface we must chose one of the two methods discussed above. The first method suggests only some guesses, while the second method has a clear algorithm for the derivation of the boundary conditions. That is why we choose here the second method.

We derive new seismic equations in each component (elastic and poroelastic) and the boundary conditions on the common boundary. For these boundary conditions the very little is known and only for the liquid filtration (see for example [10]).

For three different sets of μ_0, $\lambda_0, \dots$ for each component we derive three different mathematical models, which describe the process with different degrees of approximation.

We start with the integral identities, defining the weak solution $\mathbf{w}^\varepsilon$ and p^ε, and use the two-scale expansion method [11, 12], when we look for the solution in the form

$$\begin{gathered}\mathbf{w}^\varepsilon(\mathbf{x},t) = \mathbf{w}(\mathbf{x},t) + \mathbf{W}_0(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}) + \varepsilon\,\mathbf{W}_1(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}) + o(\varepsilon),\\ p^\varepsilon(\mathbf{x},t) = p(\mathbf{x},t) + P_0(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}) + \varepsilon\,P_1(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}) + o(\varepsilon)\end{gathered}$$

with 1-periodic in the variable $\mathbf{y}$ functions $\mathbf{W}_i(\mathbf{x},t,\mathbf{y})$, $P_i(\mathbf{x},t,\mathbf{y})$, $i = 0,1,\dots$

This method is rather heuristic and may lead to the wrong answer. But our guesses are based upon the strong theory, suggested by G. Nguetseng [13, 14]. For the rigorous derivation of seismic equations in poroelastic media, which dictate the correct two-scale expansion, see [8].

Finally, to calculate limits as $\varepsilon \to 0$ in corresponding integral identities, we apply the well-known result

$$\lim_{\varepsilon\to 0}\int_\Omega F(\mathbf{x},\frac{\mathbf{x}}{\varepsilon},t)\,dx\,dt = \int_\Omega\Big(\int_Y F(\mathbf{x},\mathbf{y},t)dy\Big)\,dx\,dt \tag{1.9}$$

for any smooth 1-periodic in the variable $\mathbf{y}\in Y$ function $F(\mathbf{x},\mathbf{y},t)$.

2. Statement of the problem

For the sake of simplicity we suppose that $Q = \{\mathbf{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 > 0\}$, $\Omega^{(0)} = \{\mathbf{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : 0 < x_3 < H\}$, $\Omega = \{\mathbf{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 > H\}$, $\mathbf{F} = 0$, and

$$\alpha_{p,f} = \bar{c}_f^2,\ \ \alpha_{p,s} = \bar{c}_s^2.$$

Let Y be a unit cube in $\mathbb{R}^3$, $Y = Y_f\cup\gamma\cup Y_s$. We assume that pore space Ω_f^ε is a periodic repetition in Ω of the elementary cell εY_f, the solid skeleton Ω_s^ε is a periodic repetition in Ω of the elementary cell εY_s, and the boundary Γ^ε between a pore space and a solid skeleton is a periodic repetition in Ω of the boundary $\varepsilon\gamma$. Detailed description of the sets Y_f and Y_s is done in [8]. From these suppositions,

$$\chi_0(\mathbf{x}) = \chi^\varepsilon(\mathbf{x}) = \chi(\frac{\mathbf{x}}{\varepsilon}),$$

where $\chi(\mathbf{y})$ is a 1-periodic function such that $\chi(\mathbf{y}) = 1$ for $\mathbf{y}\in Y_f$ and $\chi(\mathbf{y}) = 0$ for $\mathbf{y}\in Y_s$.

For a fixed $\varepsilon > 0$ the displacement vector $\mathbf{w}^\varepsilon$ and pressure p^ε satisfy Lame's system

$$\varrho_s^{(0)}\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2} = \nabla\cdot\mathbb{P}_s^{(0)}, \quad p^\varepsilon + \bar{c}_{s,0}^2\nabla\cdot\mathbf{w}^\varepsilon = 0, \tag{2.1}$$

$$\mathbb{P}_s^{(0)} = \alpha_\lambda^{(0)}\,\mathbb{D}(x,\mathbf{w}^\varepsilon) - p^\varepsilon\mathbb{I} \tag{2.2}$$

in the domain $\Omega^{(0)}$ for $t > 0$, and the system (1.1)-(1.3) with $\chi_0 = \chi^\varepsilon(\mathbf{x})$, $\varrho = \varrho^\varepsilon = \varrho_f\,\chi^\varepsilon + \varrho_s(1-\chi^\varepsilon)$, and $\alpha_p = \alpha_p^\varepsilon = \alpha_{p,f}\chi^\varepsilon + \alpha_{p,s}(1-\chi^\varepsilon)$ in the domain Ω for $t > 0$.

On the common boundary $S^{(0)} = \{\mathbf{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 = H\}$ the displacement vector and normal tensions are continuous:

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^{(0)}}} \mathbf{w}^\varepsilon(\mathbf{x},t) = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} \mathbf{w}^\varepsilon(\mathbf{x},t), \quad \mathbf{x}^0\in S^{(0)}, \tag{2.3}$$

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^{(0)}}} \mathbb{P}^{(0)}(\mathbf{x},t)\cdot\mathbf{e}_3 = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} \mathbb{P}(\mathbf{x},t)\cdot\mathbf{e}_3, \quad ,\mathbf{x}^0\in S^{(0)}, \tag{2.4}$$

where $\mathbf{e}_3 = (0,0,1)$.

The problem is complemented with the boundary condition

$$\mathbb{P}_s^{(0)}\cdot\mathbf{e}_3 = -p^0(\mathbf{x}',t)\mathbf{e}_3, \quad \mathbf{x}' = (x_1,x_2) \tag{2.5}$$

on the outer boundary $S = \{\mathbf{x} = (x_1,x_2,x_3)\in\mathbb{R}^3 : x_3 = 0\}$ for $t > 0$ and homogeneous initial conditions

$$\mathbf{w}^\varepsilon(\mathbf{x},0) = \frac{\partial\mathbf{w}^\varepsilon}{\partial t}(\mathbf{x},0) = 0. \tag{2.6}$$

Let $\varsigma(\mathbf{x})$ be the characteristic function of the domain Ω and

$$\tilde{\varrho}^\varepsilon = (1-\varsigma)\varrho_s^{(0)} + \varsigma\,\varrho^\varepsilon,\ \ \tilde{\alpha}_p^\varepsilon = (1-\varsigma)\bar{c}_{s,0}^2 + \varsigma\,\alpha_p^\varepsilon.$$

Then the above formulated problem takes the form

$$\tilde{\varrho}^\varepsilon\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2} = \nabla\cdot\big((1-\varsigma)\mathbb{P}_s^{(0)} + \varsigma\mathbb{P}\big), \tag{2.7}$$

$$p^\varepsilon + \tilde{\alpha}_p^\varepsilon\nabla\cdot\mathbf{w}^\varepsilon = 0, \tag{2.8}$$

$$\mathbb{P} = \chi^\varepsilon\,\alpha_\mu\mathbb{D}(x,\frac{\partial\mathbf{w}^\varepsilon}{\partial t}) + (1-\chi^\varepsilon)\alpha_\lambda\,\mathbb{D}(x,\mathbf{w}^\varepsilon) - \big(\chi^\varepsilon\frac{\alpha_\nu}{\bar{c}_f^2}\frac{\partial p^\varepsilon}{\partial t} + p^\varepsilon\big)\mathbb{I}, \tag{2.9}$$

where in (2.9) we have used the consequence of (2.8) in the form

$$\varsigma\,\chi^\varepsilon\alpha_\nu(\nabla\cdot\frac{\partial\mathbf{w}^\varepsilon}{\partial t}) = -\varsigma\,\chi^\varepsilon\frac{\alpha_\nu}{\bar{c}_f^2}\frac{\partial p^\varepsilon}{\partial t}.$$

Equation (2.7) is understood in the sense of distributions. That is, for any smooth functions φ with a compact support in $\overline{Q}$ the following integral identity

$$\int_{Q_T}\Big(\tilde{\varrho}^\varepsilon\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}\cdot\varphi + \big((1-\varsigma)\mathbb{P}_s^{(0)} + \varsigma\mathbb{P}\big):\mathbb{D}(x,\varphi) + \nabla\cdot(p^0\varphi)\Big)\,dx\,dt = 0 \tag{2.10}$$

holds. We call such solution the *weak solution*.

In (2.10) $Q_T = Q\times(0,T)$ and the convolution $\mathbb{A}:\mathbb{B}$ of two tensors $\mathbb{A} = (A_{ij})$ and $\mathbb{B} = (B_{ij})$ is defined as $\mathbb{A}:\mathbb{B} = \operatorname{tr}(\mathbb{A}\cdot\mathbb{B}) = \sum_{i,j=1}^3 A_{ij}B_{ji}$.

Using standard methods one can prove that for any positive $\varepsilon > 0$ and given smooth function p^0 there exists a unique weak solution of the problem (2.7)-(2.9) which makes sense to the integral identity (2.10). We look for the limit of the weak solutions for the following cases:

(I) $\mu_0=\lambda_0=\lambda_0^{(0)}=0$, $\mu_1=\lambda_1=\infty$, $0\leqslant\nu_0<\infty$, $\lambda_0^{(0)}=\lim_{\varepsilon\to 0}\alpha_\lambda^{(0)}$;

(II) $\mu_0=\lambda_0=\lambda_0^{(0)}=\mu_1=\nu_0=0$, $\lambda_1=\infty$;

(III) $\mu_0=\nu_0=0$, $0<\lambda_0, \lambda_0^{(0)}, \mu_1<\infty$.

3. Homogenization: case (I)

According to [9], the two-scale expansion for the weak solution of the problem (2.7)-(2.9) under conditions (**I**) has the form

$$\mathbf{w}^\varepsilon(\mathbf{x},t)=\mathbf{w}(\mathbf{x},t)+o(\varepsilon),\quad p^\varepsilon(\mathbf{x},t)=p(\mathbf{x},t)+o(\varepsilon), \tag{3.1}$$

where $\lim_{\varepsilon\to 0}o(\varepsilon)=0$.

The substitution (3.1) into (2.10) results in the integral identity

$$\begin{aligned}&\int_{\Omega_T}\Big(\big(\chi(\frac{\mathbf{x}}{\varepsilon})\varrho_f+\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\varrho_s\big)\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot\varphi-(\chi(\frac{\mathbf{x}}{\varepsilon})\frac{\alpha_\nu}{\bar{c}_f^2}\frac{\partial p}{\partial t}+p)\nabla\cdot\varphi\Big)\,dx\,dt\\&+\int_{Q_T}\nabla\cdot(p^0\varphi)\,dx\,dt+\int_{\Omega_T^{(0)}}\big(\varrho_s^{(0)}\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot\varphi-p(\nabla\cdot\varphi)\big)\,dx\,dt=o(\varepsilon).\end{aligned}\tag{3.2}$$

Now we use (1.9) and after the limit in (3.2) as $\varepsilon\to 0$ arrive at the integral identity

$$\begin{aligned}&\int_{\Omega_T}\big(\hat{\varrho}\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot\varphi-(m\frac{\nu_0}{\bar{c}_f^2}\frac{\partial p}{\partial t}+p)\nabla\cdot\varphi\big)\,dx\,dt+\int_{Q_T}\nabla\cdot(p^0\varphi)\,dx\,dt\\&+\int_{\Omega_T^{(0)}}\big(\varrho_s^{(0)}\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot\varphi-p(\nabla\cdot\varphi)\big)\,dx\,dt=0,\end{aligned}\tag{3.3}$$

where $\hat{\varrho}=m\varrho_f+(1-m)\varrho_s$ and $m=\int_Y\chi(\mathbf{y})dy$.

Next we rewrite (2.8) as

$$\Big(\frac{(1-\varsigma)}{\bar{c}_{s,0}^2}+\frac{\varsigma\,\chi(\frac{\mathbf{x}}{\varepsilon})}{\bar{c}_f^2}+\frac{\varsigma\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)}{\bar{c}_s^2}\Big)p^\varepsilon+\nabla\cdot\mathbf{w}^\varepsilon=0\,. \tag{3.4}$$

We multiply the result by a smooth function $\psi(\mathbf{x},t)$ with a compact support in Q, and integrate by parts over domain Q_T:

$$\int_{Q_T}\Big(\psi\big(\frac{(1-\varsigma)}{\bar{c}_{s,0}^2}+\frac{\varsigma\,\chi(\frac{\mathbf{x}}{\varepsilon})}{\bar{c}_f^2}+\frac{\varsigma\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)}{\bar{c}_s^2}\big)p^\varepsilon-\nabla\psi\cdot\mathbf{w}^\varepsilon\Big)\,dx\,dt=0. \tag{3.5}$$

As above, we substitute (3.1) into (3.5) and pass to the limit as $\varepsilon\to 0$:

$$\int_{Q_T}\Big(\psi\big(\frac{(1-\varsigma)}{\bar{c}_{s,0}^2}+\frac{\varsigma\,m}{\bar{c}_f^2}+\frac{\varsigma(1-m)}{\bar{c}_s^2}\big)p-\nabla\psi\cdot\mathbf{w}\Big)\,dx\,dt=0. \tag{3.6}$$

Integral identities (3.3) and (3.6), complemented with initial conditions

$$\mathbf{w}(\mathbf{x},0)=\frac{\partial\mathbf{w}}{\partial t}(\mathbf{x},0)=0, \tag{3.7}$$

form mathematical model (I) of seismic in composite media.

In fact, these identities contain the differential equations in Ω and $\Omega^{(0)}$ and the boundary conditions on S and $S^{(0)}$.

Let φ be a smooth function with a compact support in $\Omega^{(0)}$. Rewriting (3.3) as

$$\int_{\Omega_T^{(0)}} (\varrho_s^{(0)} \frac{\partial^2 \mathbf{w}}{\partial t^2} + \nabla p) \cdot \varphi \, dx \, dt = 0$$

and using the arbitrary choice of φ we conclude that

$$\varrho_s^{(0)} \frac{\partial^2 \mathbf{w}}{\partial t^2} = -\nabla p \tag{3.8}$$

in the domain $\Omega^{(0)}$ for $t > 0$.

For functions φ with a compact support in Ω, (3.3) implies

$$\hat{\varrho} \frac{\partial^2 \mathbf{w}}{\partial t^2} = -\nabla (p + m \frac{\nu_0}{\bar{c}_f^2} \frac{\partial p}{\partial t}), \quad \hat{\varrho} = m \varrho_f + (1 - m) \varrho_s \tag{3.9}$$

in the domain Ω for $t > 0$.

Now, if we choose $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with a compact support in Q and $\varphi_3(\mathbf{x}, t) \neq 0$ for $\mathbf{x} \in S^{(0)}$, then the integration by parts in (3.3) together with (3.8) and (3.9) result in

$$\int_{S_T^{(0)}} \big(p^- - (p^+ + m \frac{\nu_0}{\bar{c}_f^2} \frac{\partial p^+}{\partial t})\big) \varphi_3 \, dx \, dt = 0,$$

where

$$p^-(x_1, x_2, t) = p(x_1, x_2, H - 0, t), \quad p^+(x_1, x_2, t) = p(x_1, x_2, H + 0, t).$$

Therefore,

$$\lim_{\substack{\mathbf{x} \to \mathbf{x}^0 \\ \mathbf{x} \in \Omega^{(0)}}} p(\mathbf{x}, t) = \lim_{\substack{\mathbf{x} \to \mathbf{x}^0 \\ \mathbf{x} \in \Omega}} \big(p(\mathbf{x}, t) + m \frac{\nu_0}{\bar{c}_f^2} \frac{\partial p}{\partial t}(\mathbf{x}, t)\big), \ \ \mathbf{x}^0 \in S^{(0)}. \tag{3.10}$$

Finally, for functions φ with a compact support in $\Omega^{(0)}$ and $\varphi_3(\mathbf{x}, t) \neq 0$ for $\mathbf{x} \in S$ the integration by parts in (3.3) together with (3.8) result in

$$\int_{S_T} (p - p^0) \varphi_3 \, dx \, dt = 0,$$

or

$$p(\mathbf{x}, t) = p^0(\mathbf{x}, t), \ \ \mathbf{x} \in S. \tag{3.11}$$

In the same way as above, it can be shown that (3.6) implies continuity equations

$$\frac{1}{\bar{c}_{s,0}^2} p + \nabla \cdot \mathbf{w} = 0 \tag{3.12}$$

and

$$\big(\frac{m}{\bar{c}_f^2} + \frac{(1 - m)}{\bar{c}_s^2}\big) p + \nabla \cdot \mathbf{w} = 0 \tag{3.13}$$

in the domains $\Omega^{(0)}$ and Ω respectively, and the boundary condition

$$\lim_{\substack{\mathbf{x} \to \mathbf{x}^0 \\ \mathbf{x} \in \Omega^{(0)}}} \mathbf{e}_3 \cdot \mathbf{w}(\mathbf{x}, t) = \lim_{\substack{\mathbf{x} \to \mathbf{x}^0 \\ \mathbf{x} \in \Omega}} \mathbf{e}_3 \cdot \mathbf{w}(\mathbf{x}, t), \quad \mathbf{x}^0 \in S^{(0)} \tag{3.14}$$

on the common boundary $S^{(0)}$.

Differential equations (3.8), (3.9), (3.12), and (3.13), boundary conditions (3.10), (3.11), and (3.14), and initial conditions (3.7) constitute the mathematical model (I) in its differential form.

4. Homogenization: case (II)

For this case we put

$$\begin{gathered}\mathbf{w}^{\varepsilon}(\mathbf{x},t)=(1-\varsigma)\mathbf{w}(\mathbf{x},t)+\varsigma\chi(\frac{\mathbf{x}}{\varepsilon})\mathbf{w}^{(f,\varepsilon)}(\mathbf{x},t)+\varsigma\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\mathbf{w}_s(\mathbf{x},t)+o(\varepsilon),\\ p^{\varepsilon}(\mathbf{x},t)=p(\mathbf{x},t)+o(\varepsilon),\end{gathered}\tag{4.1}$$

where

$$\mathbf{w}^{(f,\varepsilon)}(\mathbf{x},t)=\mathbf{W}^{(f)}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}),$$

and $\mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y})$ is a 1-periodic in the variable $\mathbf{y}$ function.

The substitution (4.1) into (2.10) results in the integral identity

$$\begin{aligned}&\int_{Q_T}\nabla\cdot(p^0\varphi)\,dx\,dt+\int_{\Omega_T^{(0)}}\big(\varrho_s^{(0)}\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot\varphi-p(\nabla\cdot\varphi)\big)\,dx\,dt\\&+\int_{\Omega_T}\Big(\big(\varrho_f\chi(\frac{\mathbf{x}}{\varepsilon})\frac{\partial^2\mathbf{W}^{(f)}}{\partial t^2}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})+\varrho_s\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\frac{\partial^2\mathbf{w}_s}{\partial t^2}\big)\cdot\varphi-p(\nabla\cdot\varphi)\Big)\,dx\,dt\\&=-\int_{\Omega_T}\alpha_\mu\,\chi(\frac{\mathbf{x}}{\varepsilon})\mathbb{D}(x,\frac{\partial\mathbf{w}^{(f,\varepsilon)}}{\partial t}):\mathbb{D}(x,\varphi)\,dx\,dt+o(\varepsilon),\end{aligned}\tag{4.2}$$

which holds for any smooth function $\varphi(\mathbf{x},t)$. Let

$$\mathbf{w}^{(f)}(\mathbf{x},t)=\varsigma\lim_{\varepsilon\to0}\chi(\frac{\mathbf{x}}{\varepsilon})\mathbf{W}^{(f)}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})=\varsigma\int_{Y_f}\mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y})dy$$

be the weak limit of the sequence $\{\mathbf{w}^{\varepsilon}\}$. Then after the limit as $\varepsilon\to0$ we arrive at the integral identity

$$\begin{aligned}&\int_{Q_T}\nabla\cdot(p^0\varphi)\,dx\,dt+\int_{\Omega_T^{(0)}}\big(\varrho_s^{(0)}\frac{\partial^2\mathbf{w}}{\partial t^2}\cdot\varphi-p(\nabla\cdot\varphi)\big)\,dx\,dt\\&=\int_{\Omega_T}\Big(\big(\varrho_f\frac{\partial^2\mathbf{w}^{(f)}}{\partial t^2}+\varrho_s(1-m)\frac{\partial^2\mathbf{w}_s}{\partial t^2}\big)\cdot\varphi-p(\nabla\cdot\varphi)\Big)\,dx\,dt=0.\end{aligned}\tag{4.3}$$

Note that the term $\alpha_\mu\mathbb{D}(x,\frac{\partial\mathbf{w}^{(f,\varepsilon)}}{\partial t})$ in the right-hand side of (4.2) converges to zero due to the supposition $\lim_{\varepsilon\searrow0}\alpha_\mu=\lim_{\varepsilon\searrow0}\frac{\alpha_\mu}{\varepsilon}=0$:

$$\alpha_\mu\mathbb{D}\big(x,\frac{\partial\mathbf{w}^{(f,\varepsilon)}}{\partial t}(\mathbf{x},t)\big)=\alpha_\mu\mathbb{D}\big(x,\frac{\partial\mathbf{W}^{(f)}}{\partial t}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})\big)+\frac{\alpha_\mu}{\varepsilon}\mathbb{D}\big(y,\frac{\partial\mathbf{W}^{(f)}}{\partial t}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})\big).$$

The substitution of (4.1) into the continuity equation (2.8) leads to the integral identity

$$\begin{aligned}&\int_{Q_T}\Big(\psi\big(\frac{(1-\varsigma)}{\bar c_{s,0}^2}+\frac{\varsigma\,\chi^{\varepsilon}}{\bar c_f^2}+\frac{\varsigma(1-\chi^{\varepsilon})}{\bar c_s^2}\big)p\\&-\nabla\psi\cdot\big((1-\varsigma)\mathbf{w}+\varsigma\chi^{\varepsilon}\mathbf{W}^{(f)}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})+\varsigma(1-\chi^{\varepsilon})\mathbf{w}_s\big)\Big)\,dx\,dt=o(\varepsilon).\end{aligned}\tag{4.4}$$

The limit here as $\varepsilon\to0$ results in

$$\begin{gathered}\int_{Q_T}\Big(\psi\big(\frac{(1-\varsigma)}{\bar c_{s,0}^2}+\frac{\varsigma\,m}{\bar c_f^2}+\frac{\varsigma(1-m)}{\bar c_s^2}\big)p+\\ \nabla\psi\cdot\big((1-\varsigma)\mathbf{w}+\varsigma\,\mathbf{w}^{(f)}+\varsigma(1-m)\mathbf{w}_s\big)\Big)\,dx\,dt=0.\end{gathered}\tag{4.5}$$

As in previous section we conclude that integral identities (4.3) and (4.5) imply differential equations in $\Omega^{(0)}$ and Ω and boundary conditions on the boundaries $S^{(0)}$ and S.

Namely, in the domain $\Omega^{(0)}$ the displacements vector $\mathbf{w}$ and pressure p of the solid component satisfy the seismic system

$$\varrho_s^{(0)}\frac{\partial^2\mathbf{w}}{\partial t^2} = -\nabla p, \tag{4.6}$$

$$\frac{1}{\bar{c}_{s,0}^2}p + \nabla\cdot\mathbf{w} = 0. \tag{4.7}$$

In the domain Ω the displacements vector $\mathbf{w}_s$ of the solid component, displacements vector $\mathbf{w}^{(f)}$ of the liquid component, and pressure p of the medium satisfy the seismic system

$$\varrho_f\frac{\partial^2\mathbf{w}^{(f)}}{\partial t^2} + \varrho_s(1-m)\frac{\partial^2\mathbf{w}_s}{\partial t^2} = -\nabla p, \tag{4.8}$$

$$\Big(\frac{m}{\bar{c}_f^2} + \frac{(1-m)}{\bar{c}_s^2}\Big)p + \nabla\cdot\big(\mathbf{w}^{(f)} + (1-m)\mathbf{w}_s\big) = 0. \tag{4.9}$$

On the common boundary $S^{(0)}$ the displacements vectors $\mathbf{w}$, $\mathbf{w}_s$, and $\mathbf{w}^{(f)}$ and pressure p satisfy continuity conditions

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^{(0)}}} p(\mathbf{x},t) = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} p(\mathbf{x},t), \tag{4.10}$$

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^{(0)}}} \mathbf{e}_3\cdot\mathbf{w}(\mathbf{x},t) = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} \mathbf{e}_3\cdot\big(\mathbf{w}^{(f)}(\mathbf{x},t) + (1-m)\mathbf{w}_s(\mathbf{x},t)\big). \tag{4.11}$$

Finally, on the outer boundary S,

$$p(\mathbf{x},t) = p^0(\mathbf{x},t). \tag{4.12}$$

As above, we have to add the initial conditions:

$$\begin{aligned}\mathbf{w}(\mathbf{x},0) = \frac{\partial\mathbf{w}}{\partial t}(\mathbf{x},0) = \mathbf{w}_s(\mathbf{x},0) = \frac{\partial\mathbf{w}_s}{\partial t}(\mathbf{x},0)\\ = \mathbf{w}^{(f)}(\mathbf{x},0) = \frac{\partial\mathbf{w}^{(f)}}{\partial t}(\mathbf{x},0) = 0.\end{aligned} \tag{4.13}$$

The obtained system of differential equations and boundary and initial conditions is still incomplete. We have no differential equation for the liquid displacements $\mathbf{w}^{(f)}$. To find the missing equation we pass to the limit $\varepsilon\to 0$ in (4.2) with test functions φ^ε in the form

$$\varphi^\varepsilon(\mathbf{x},t) = h(\mathbf{x},t)\varphi_0(\frac{\mathbf{x}}{\varepsilon}),$$

where h is a smooth function with a compact support in Ω and $\varphi_0(\mathbf{y})$ is a smooth function with a compact support in Y_f (that is φ^ε vanishes outside of the pore space Ω_f).

For an arbitrary function $\varphi_0(\mathbf{y})$ the term $p\nabla\cdot\varphi^\varepsilon$ becomes unbounded as $\varepsilon\to 0$:

$$\nabla\cdot\varphi^\varepsilon = \big(\nabla_x h(\mathbf{x},t)\big)\cdot\varphi_0(\frac{\mathbf{x}}{\varepsilon}) + \frac{1}{\varepsilon}h(\mathbf{x},t)\big(\nabla_y\cdot\varphi_0(\frac{\mathbf{x}}{\varepsilon})\big).$$

Therefore, we require that conditions

$$\nabla_y\cdot\varphi_0(\mathbf{y}) = 0, \quad \mathbf{y}\in Y_f, \tag{4.14}$$

$$\varphi_0(\mathbf{y}) = 0, \quad \mathbf{y} \in \gamma \tag{4.15}$$

hold.

The term $\alpha_\mu \mathbb{D}(x, \frac{\partial \mathbf{w}^{(f,\varepsilon)}}{\partial t}) : \mathbb{D}(x, \varphi^\varepsilon)$ in the right-hand side of (4.2) converges to zero because of the assumptions $\lim_{\varepsilon\searrow 0} \alpha_\mu = \lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon} = \lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = 0$:

$$\begin{aligned}
&\alpha_\mu \mathbb{D}\big(x, \frac{\partial \mathbf{w}^{(f,\varepsilon)}}{\partial t}(\mathbf{x},t)\big) : \mathbb{D}(x, \varphi^\varepsilon) \\
&= \alpha_\mu \Big(\mathbb{D}\big(x, \frac{\partial \mathbf{W}^{(f)}}{\partial t}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})\big) + \frac{1}{\varepsilon}\mathbb{D}\big(y, \frac{\partial \mathbf{W}^{(f)}}{\partial t}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})\big)\Big) : \\
&\quad \Big(\frac{1}{2}\big((\nabla h)\otimes \varphi_0 + \varphi_0 \otimes (\nabla h)\big) + \frac{h}{\varepsilon}\mathbb{D}\big(y, \varphi_0(\frac{\mathbf{x}}{\varepsilon})\big)\Big) = o(\varepsilon).
\end{aligned}$$

Here a matrix $\mathbf{a}\otimes\mathbf{b}$ is defined as

$$(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}),$$

for any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Thus, the limit as $\varepsilon \to 0$ in (4.2) results in the integral identity

$$\begin{aligned}
&\int_{\Omega_T} \Big(\int_{Y_f} \big(\varrho_f \frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2} h - p\,(\nabla \cdot h)\big)\cdot \varphi_0 dy \Big)\, dx\, dt \\
&= \int_{\Omega_T} h(\mathbf{x},t) \Big(\int_{Y_f} \big(\varrho_f \frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2} + \nabla p\big)\cdot \varphi_0 dy \Big)\, dx\, dt = 0,
\end{aligned} \tag{4.16}$$

which holds for any smooth function $h(\mathbf{x},t)$ with a compact support in Ω and for any smooth solenoidal function $\varphi_0(\mathbf{y})$ with a compact support in Y_f.

By arbitrary choice of $h(\mathbf{x},t)$, (4.16) implies

$$\int_{Y_f} \big(\varrho_f \frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2} + \nabla p\big)\cdot \varphi_0 dy = 0. \tag{4.17}$$

This identity means that the function $\big(\varrho_f \frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2} + \nabla p\big)$ is orthogonal to any solenoidal function. Therefore there exists some 1-periodic in the variable $\mathbf{y}$ function $\Pi(\mathbf{x},t,\mathbf{y})$ such that

$$\varrho_f \frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2} + \nabla p = -\nabla_y \Pi \tag{4.18}$$

in the domain Y_f for any parameters $(\mathbf{x},t) \in \Omega_T$.

There is one equation (4.18) for two unknown functions $\mathbf{W}^{(f)}$ and Π. To derive the second equation we put in (4.4) $\psi = \varepsilon\, h(\mathbf{x},t)\, \psi_0(\frac{\mathbf{x}}{\varepsilon})$ with arbitrary smooth function $h(\mathbf{x},t)$ and arbitrary smooth 1-periodic function $\psi_0(\mathbf{y})$ and pass to the limit as $\varepsilon \to 0$:

$$\int_{\Omega_T} h(\mathbf{x},t) \Big(\int_Y \chi(\mathbf{y}) \nabla \psi_0(\mathbf{y}) \cdot \mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y}) dy \Big) dx = 0.$$

After reintegration we obtain the desired microscopic continuity equation

$$\nabla \cdot \mathbf{W}^{(f)} = 0, \quad \mathbf{y} \in Y_f. \tag{4.19}$$

A rigorous theory (see [13, 8, 9]) supplies the system (4.18), (4.19) with the boundary condition

$$\big(\mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y}) - \mathbf{w}_s(\mathbf{x},t)\big)\cdot \mathbf{n}(\mathbf{y}) = 0 \tag{4.20}$$

on the boundary γ with the unit normal $\mathbf{n}(\mathbf{y})$, and the homogeneous initial conditions

$$\mathbf{W}^{(f)}(\mathbf{x},0,\mathbf{y}) = \frac{\partial \mathbf{W}^{(f)}}{\partial t}(\mathbf{x},0,\mathbf{y}) = 0. \tag{4.21}$$

Problem (4.18)-(4.21) has been solved in [9]:

$$\varrho_f \frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2} = \varrho_f \frac{\partial^2 \underset{\approx}{\prec}_s}{\partial t^2} - \big(\mathbb{I} - \sum_{i=1}^{3} \nabla_y \Pi_i \otimes \mathbf{e}_i\big) \cdot \big(\nabla p + \varrho_f \frac{\partial^2 \underset{\approx}{\prec}_s}{\partial t^2}\big), \tag{4.22}$$

where $\Pi_i(\mathbf{y})$, $i = 1,2,3$ are solutions to the periodic boundary value problems

$$\triangle_y \Pi_i = 0, \ \mathbf{y} \in Y_f, \ (\nabla_y \Pi_i - \mathbf{e}_i) \cdot \mathbf{n}(\mathbf{y}) = 0, \quad \mathbf{y} \in \gamma.$$

Thus,

$$\varrho_f \frac{\partial^2 \mathbf{w}^{(f)}}{\partial t^2} = m\varrho_f \frac{\partial^2 \underset{\approx}{\prec}_s}{\partial t^2} - \mathbb{B}_2^{(f)} \cdot \big(\nabla p + \varrho_f \frac{\partial^2 \underset{\approx}{\prec}_s}{\partial t^2}\big), \tag{4.23}$$

where

$$\mathbb{B}_2^{(f)} = m\mathbb{I} - \sum_{i=1}^{3} \int_{Y_f} \nabla_y \Pi_i^{(f)} dy \otimes \mathbf{e}_i. \tag{4.24}$$

Differential equations (4.6)-(4.9), (4.23), boundary conditions (4.10)-(4.12), and initial conditions (4.13) form the mathematical model (II) of seismics in composite media.

5. Homogenization: case (III)

According to [8] the set of criteria (III) dictates the form of the two-scale expansion:

$$\begin{aligned} \mathbf{w}^{\varepsilon}(\mathbf{x},t) &= (1-\varsigma)\mathbf{w}(\mathbf{x},t) + \varsigma\,\chi(\frac{\mathbf{x}}{\varepsilon})\mathbf{w}^{(f,\varepsilon)}(\mathbf{x},t) + \varsigma\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\big(\mathbf{w}_s(\mathbf{x},t) \\ &\quad + \varepsilon \mathbf{w}_s^{\varepsilon}(\mathbf{x},t)\big) + o(\varepsilon), \\ p^{\varepsilon}(\mathbf{x},t) &= (1-\varsigma)p(\mathbf{x},t) + \varsigma\,\chi(\frac{\mathbf{x}}{\varepsilon})p_f(\mathbf{x},t) + \varsigma\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)p_s^{\varepsilon}(\mathbf{x},t) + o(\varepsilon), \end{aligned} \tag{5.1}$$

where

$$\mathbf{w}^{(f,\varepsilon)}(\mathbf{x},t) = \mathbf{W}^{(f)}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}), \ \ \mathbf{w}_s^{\varepsilon}(\mathbf{x},t) = \mathbf{W}_s(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}), \ \ p_s^{\varepsilon}(\mathbf{x},t) = P_s(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}),$$

and $\mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y})$, $\mathbf{W}_s(\mathbf{x},t,\mathbf{y})$, $P_s(\mathbf{x},t,\mathbf{y})$ are 1-periodic in the variable $\mathbf{y}$ functions.

Next we express the pressure p_s^{ε} in the solid component in Ω using the continuity equation (2.8) and two-scale expansion (5.1):

$$p_s^{\varepsilon}(\mathbf{x},t) = -\bar{c}_s^2\big(\nabla \cdot \mathbf{w}_s(\mathbf{x},t) + \nabla_y \cdot \mathbf{W}_s(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})\big) + o(\varepsilon). \tag{5.2}$$

The substitution of (5.1) and (5.2) into (2.10) results in the integral equality

$$\begin{aligned}
&\int_{Q_T} \nabla\cdot(p^0\varphi)\,dx\,dt + \int_{\Omega_T^{(0)}} \big(\varrho_s^{(0)}\frac{\partial^2 \mathbf{w}}{\partial t^2}\cdot\varphi + \big(\lambda_0^{(0)}\mathbb{D}(x,\mathbf{w}) - p\mathbb{I}\big) : \mathbb{D}(x,\varphi)\big)\,dx\,dt\\
&+ \int_{\Omega_T} \Big(\big(\varrho_f\chi(\frac{\mathbf{x}}{\varepsilon})\frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}) + \varrho_s\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\frac{\partial^2 \mathbf{w}_s}{\partial t^2}\big)\cdot\varphi\Big)\,dx\,dt\\
&+ \int_{\Omega_T} \lambda_0\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\Big(\mathfrak{N}^{(0)} : \big(\mathbb{D}(x,\mathbf{w}_s) + \mathbb{D}\big(y,\mathbf{W}_s(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})\big)\big)\Big) : \mathbb{D}(x,\varphi)\,dx\,dt\\
&= -\int_{\Omega_T} \chi(\frac{\mathbf{x}}{\varepsilon})\Big(\frac{\alpha_\mu}{\varepsilon}\mathbb{D}\big(y,\frac{\partial \mathbf{W}^{(f)}}{\partial t}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})\big) - p_f\mathbb{I}\Big) : \mathbb{D}(x,\varphi)\,dx\,dt + o(\varepsilon),
\end{aligned} \tag{5.3}$$

which holds for any smooth function $\varphi(\mathbf{x},t)$. In (5.3)

$$\mathfrak{N}^{(0)} = \sum_{i,j=1}^{3} \mathbb{J}^{ij}\otimes\mathbb{J}^{ij} + \frac{\bar{c}_s^2}{\lambda_0}\mathbb{I}\otimes\mathbb{I},\quad \mathbb{J}^{ij} = \frac{1}{2}\big(\mathbf{e}_i\otimes\mathbf{e}_j + \mathbf{e}_j\otimes\mathbf{e}_i\big),$$

$\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a standard Cartesian basis, and the fourth-rank tensor $\mathbb{A}\otimes\mathbb{B}$ is the tensor (direct) product of the second-rank tensors $\mathbb{A}$ and $\mathbb{B}$:

$$(\mathbb{A}\otimes\mathbb{B}) : \mathbb{C} = \mathbb{A}(\mathbb{B} : \mathbb{C})$$

for any second-rank tensor $\mathbb{C}$.

After the limit in (5.3) as $\varepsilon\to 0$ we arrive at the integral identity

$$\begin{aligned}
&\int_{\Omega_T^{(0)}} \big(\varrho_s^{(0)}\frac{\partial^2 \mathbf{w}}{\partial t^2}\cdot\varphi + \big(\lambda_0^{(0)}\mathbb{D}(x,\mathbf{w}) - p\mathbb{I}\big) : \mathbb{D}(x,\varphi)\big)\,dx\,dt\\
&+ \int_{Q_T} \nabla\cdot(p^0\varphi)\,dx\,dt + \int_{\Omega_T} \Big(\big(\varrho_f\frac{\partial^2 \mathbf{w}^{(f)}}{\partial t^2} + \varrho_s(1-m)\frac{\partial^2 \mathbf{w}_s}{\partial t^2}\big)\cdot\varphi\Big)\,dx\,dt\\
&+ \int_{\Omega_T} \lambda_0\Big(\mathfrak{N}^{(0)} : \big((1-m)\mathbb{D}(x,\mathbf{w}_s) + \langle\mathbb{D}(y,\mathbf{W}_s)\rangle_{Y_s}\big)\Big) : \mathbb{D}(x,\varphi)\,dx\,dt\\
&= \int_{\Omega_T} \big(mp_f\mathbb{I}\big) : \mathbb{D}(x,\varphi)\,dx\,dt,
\end{aligned} \tag{5.4}$$

where

$$\mathbf{w}^{(f)} = \langle\,\mathbf{W}^{(f)}\,\rangle_{Y_f},\quad \langle F\rangle_A = \int_A F(\mathbf{y})dy,\quad A\subseteq Y.$$

To pass to the limit as $\varepsilon\to 0$ in the continuity equation (2.8) we rewrite it as an integral identity and use the representation (5.1):

$$\begin{aligned}
&\int_{Q_T} \psi\,\Big(\frac{(1-\varsigma)}{\bar{c}_{s,0}^2}p + \varsigma\,\chi(\frac{\mathbf{x}}{\varepsilon})\frac{p_f}{\bar{c}_f^2} + \varsigma\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\frac{p_s^\varepsilon}{\bar{c}_s^2}\big)\Big)\,dx\,dt\\
&-\int_{Q_T} \nabla\,\psi\cdot\big((1-\varsigma)\mathbf{w} + \varsigma\,\chi(\frac{\mathbf{x}}{\varepsilon})\mathbf{W}_f + \varsigma\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\mathbf{w}_s\big)\,dx\,dt = o(\varepsilon).
\end{aligned} \tag{5.5}$$

In the limit as $\varepsilon\to 0$ results in the integral equality

$$\begin{aligned}
&\int_{Q_T} \psi\big(\frac{(1-\varsigma)}{\bar{c}_{s,0}^2}p + \frac{\varsigma\,m}{\bar{c}_f^2}p_f + \frac{\varsigma}{\bar{c}_s^2}\,\langle P_s\rangle_{Y_s}\big)\big)\,dx\,dt\\
&-\int_{Q_T} \nabla\,\psi\cdot\big((1-\varsigma)\mathbf{w} + \varsigma\,\mathbf{w}^{(f)} + \varsigma(1-m)\mathbf{w}_s\big)\,dx\,dt = 0,
\end{aligned} \tag{5.6}$$

which holds for any smooth function ψ vanishing at S.

Finally, we rewrite the continuity equation in the pore space Ω_f as the corresponding integral identity

$$\begin{aligned}
&\int_{\Omega_T} \psi\big(\frac{\chi^\varepsilon}{\bar{c}_f^2} p^\varepsilon + \chi^\varepsilon \nabla\cdot\mathbf{w}^\varepsilon\big)\,dx\,dt \\
&= \int_{\Omega_T} \psi\big(\frac{\chi^\varepsilon}{\bar{c}_f^2} p^\varepsilon + \nabla\cdot\mathbf{w}^\varepsilon - (1-\chi^\varepsilon)\,\nabla\cdot\mathbf{w}^\varepsilon\big)\,dx\,dt \\
&= \int_{\Omega_T} \big(\psi\,\frac{\chi^\varepsilon}{\bar{c}_f^2} p^\varepsilon - (\nabla\psi)\cdot\mathbf{w}^\varepsilon - \psi(1-\chi^\varepsilon)\nabla\cdot\mathbf{w}^\varepsilon\big)\,dx\,dt,
\end{aligned}$$

and apply the two-scale expansion (5.1):

$$\begin{aligned}
&\int_{\Omega_T} \Big(\frac{\psi}{\bar{c}_f^2}\,\chi(\frac{\mathbf{x}}{\varepsilon})p_f - (\nabla\psi)\cdot\big(\chi(\frac{\mathbf{x}}{\varepsilon})\mathbf{W}_f^\varepsilon + \big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\mathbf{w}_s\big) \\
&- \psi\big(1-\chi(\frac{\mathbf{x}}{\varepsilon})\big)\big(\nabla\cdot\mathbf{w}_s + \nabla_y\cdot\mathbf{W}_s\big)\Big)\,dx\,dt = o(\varepsilon).
\end{aligned} \tag{5.7}$$

In the limit as $\varepsilon\to 0$ results in the desired integral equality

$$\int_{\Omega_T} \big(\psi\,\frac{m}{\bar{c}_f^2}p_f - \nabla\,\psi\cdot\mathbf{w}^{(f)} - \psi\,\langle\nabla_y\cdot\mathbf{W}_s\rangle_{Y_s}\big)\,dx\,dt = 0. \tag{5.8}$$

The localization of (5.4), (5.6), and (5.8) gives as the Lame system

$$\varrho_s^{(0)}\frac{\partial^2\mathbf{w}}{\partial t^2} = \nabla\cdot\mathbb{P}_s^{(0)}, \quad \mathbb{P}_s^{(0)} = \lambda_0^{(0)}\,\mathbb{D}(x,\mathbf{w}) - p\mathbb{I}, \tag{5.9}$$

$$p + \bar{c}_{s,0}^2\nabla\cdot\mathbf{w} = 0 \tag{5.10}$$

in the domain $\Omega^{(0)}$ for $t>0$, the macroscopic dynamic equation

$$\varrho_f\frac{\partial^2\mathbf{w}^{(f)}}{\partial t^2} + \varrho_s(1-m)\frac{\partial^2\mathbf{w}_s}{\partial t^2} = \nabla\cdot\widehat{\mathbb{P}}, \tag{5.11}$$

$$\widehat{\mathbb{P}} = \lambda_0\,\mathfrak{N}^{(0)} : \big((1-m)\mathbb{D}(x,\mathbf{w}_s) + \langle\mathbb{D}(y,\mathbf{W}_s)\rangle_{Y_s}\big) - mp_f\mathbb{I} \tag{5.12}$$

for the solid component and the macroscopic continuity equation

$$\frac{m}{\bar{c}_f^2}p_f + \nabla\cdot\mathbf{w}^{(f)} = \langle\nabla_y\cdot\mathbf{W}_s\rangle_{Y_s} \tag{5.13}$$

for the liquid component in the domain Ω for $t>0$.

The same localization of (5.4) and (5.6) also provides the boundary condition

$$\mathbb{P}_s^{(0)}\cdot\mathbf{e}_3 = p^0\cdot\mathbf{e}_3 \tag{5.14}$$

on the outer boundary S with the unit normal $\mathbf{e}_3$, and the continuity conditions

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^{(0)}}} \mathbb{P}^{(0)}(\mathbf{x},t)\cdot\mathbf{e}_3 = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} \widehat{\mathbb{P}}(\mathbf{x},t)\cdot\mathbf{e}_3, \tag{5.15}$$

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^{(0)}}} \mathbf{w}(\mathbf{x},t)\cdot\mathbf{e}_3 = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} \big(\mathbf{w}^{(f)}(\mathbf{x},t) + (1-m)\mathbf{w}_s(\mathbf{x},t)\big)\cdot\mathbf{e}_3 \tag{5.16}$$

on the common boundary $S^{(0)}\ni\mathbf{x}^0$ with the unit normal $\mathbf{e}_3$.

More detailed mathematical analysis shows that

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^{(0)}}} \mathbf{w}(\mathbf{x},t) = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} (1-m)\mathbf{w}_s(\mathbf{x},t) \tag{5.17}$$

for $\mathbf{x}^0 \in S^{(0)}$. Unfortunately we have no possibility to prove the statement due technical reasons.

Differential equations and boundary conditions are supplemented with initial conditions

$$
\begin{aligned}
\mathbf{w}(\mathbf{x},0) &= \frac{\partial \mathbf{w}}{\partial t}(\mathbf{x},0) = \mathbf{w}_s(\mathbf{x},0) = \frac{\partial \mathbf{w}_s}{\partial t}(\mathbf{x},0) \\
&= \mathbf{w}^{(f)}(\mathbf{x},0) = \frac{\partial \mathbf{w}^{(f)}}{\partial t}(\mathbf{x},0) = 0.
\end{aligned} \tag{5.18}
$$

However, the obtained system (5.9)-(5.18) is still incomplete. We need two more differential equations for $\mathbf{W}_s$ and $\mathbf{W}^{(f)}$. More precisely, we have to express the terms $\langle \mathbb{D}(y,\mathbf{W}_s)\rangle_{Y_s}$ and $\langle \nabla_y \cdot \mathbf{W}_s\rangle_{Y_s}$ by means of functions $\mathbb{D}(x,\mathbf{w}_s)$ and p_f and rewrite (5.12) and (5.13) as

$$
\widehat{\mathbb{P}} = \lambda_0\, \mathfrak{N}^s : \mathbb{D}(x,\mathbf{w}_s) - p_f \mathbb{C}^s, \tag{5.19}
$$

$$
\frac{m}{\bar{c}_f^2} p_f + \nabla\cdot \mathbf{w}^{(f)} = \mathbb{C}_0^s : \mathbb{D}(x, \precapprox_s) + \frac{c_0^s}{\lambda_0} p_f. \tag{5.20}
$$

To find the missing equation for the function $\mathbf{W}_s$ let us consider the integral identity (5.3). As in previous section, we choose test functions in the form $\varphi^\varepsilon(\mathbf{x},t) = \varepsilon\, h(\mathbf{x},t)\, \varphi_0(\frac{\mathbf{x}}{\varepsilon})$, where h is an arbitrary smooth function with a compact support in Ω vanishing on S, and $\varphi_0(\mathbf{y})$ is an arbitrary 1-periodic smooth function with a compact support in Y_s.

The limit in (5.3) as $\varepsilon \to 0$ with chosen test functions results in

$$
\begin{aligned}
&\int_{\Omega_T} h \Big(\int_Y \Big(\lambda_0 \big(1-\chi(\mathbf{y})\big) \big(\mathfrak{N}^{(0)} : (\mathbb{D}(x,\mathbf{w}_s) \\
&+ \mathbb{D}(y,\mathbf{W}_s)\big) - \chi(\mathbf{y})\, m p_f \mathbb{I} \Big) : \mathbb{D}(y,\varphi_0) dy \Big)\, dx\, dt = 0
\end{aligned}
$$

After a localization we obtain the differential equation

$$
\nabla_y \cdot \Big(\lambda_0 \big(1-\chi(\mathbf{y})\big) \mathfrak{N}^{(0)} : \big(\mathbb{D}(x,\mathbf{w}_s) + \mathbb{D}(y,\mathbf{W}_s)\big) - m p_f\, \chi(\mathbf{y}) \Big) = 0 \tag{5.21}
$$

in the domain Y, which is understood in the sense of distributions. That is, as a usual differential equation

$$
\nabla_y \cdot \Big(\mathfrak{N}^{(0)} : \big(\mathbb{D}(x,\mathbf{w}_s) + \mathbb{D}(y,\mathbf{W}_s)\big) \Big) = 0 \tag{5.22}
$$

in the domain Y_s. In the same way using test functions with a compact support localizes at γ we derive the boundary condition

$$
\Big(\lambda_0 \mathfrak{N}^{(0)} : \big(\mathbb{D}(x,\mathbf{w}_s) + \mathbb{D}(y,\mathbf{W}_s)\big) \Big) \cdot \mathbf{n} = - m p_f\, \mathbf{n} \tag{5.23}
$$

on the boundary γ. Here $\mathbf{n}$ is a unit normal to γ.

The problem (5.18), (5.19) is completed with the periodicity conditions on the remaining part $\partial Y_s \backslash \gamma$ of the boundary ∂Y_s.

Let $\mathbf{U}_2^{(ij)}(\mathbf{y})$ and $\mathbf{U}_2^{(0)}(\mathbf{y})$ be solutions of periodic problems

$$
\nabla_y \cdot \Big((1-\chi) \Big(\mathfrak{N}^{(0)} : \big(\mathbb{J}^{(ij)} + \mathbb{D}(y, \mathbf{U}_2^{(ij)}) \big) \Big) \Big) = 0, \tag{5.24}
$$

$$
\nabla_y \cdot \Big((1-\chi) \big(\mathfrak{N}^{(0)} : \mathbb{D}(y, \mathbf{U}_2^{(0)}) + \mathbb{I} \big) \Big) = 0 \tag{5.25}
$$

in Y. Then

$$\mathbf{W}_s(\mathbf{x},t,\mathbf{y}) = \sum_{i,j=1}^{3} \mathbf{U}_2^{(ij)}(\mathbf{y}) D_{ij}(\mathbf{x},t) + \frac{m}{\lambda_0} p_f(\mathbf{x},t)\, \mathbf{U}_2^{(0)}(\mathbf{y}),$$

where

$$D_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad \precapprox_s = (u_1,\, u_2,\, u_3), \quad \mathbb{D}(x,\mathbf{w}_s) = \sum_{i,j=1}^{3} D_{ij}\mathbb{J}^{(ij)}.$$

Thus

$$\begin{aligned}
&\langle \mathbb{D}(y,\mathbf{W}_s)\rangle_{Y_s} \\
&= \sum_{i,j=1}^{3} \langle \mathbb{D}(y,\mathbf{U}_2^{(ij)})\rangle_{Y_s} D_{ij} + \frac{m}{\lambda_0} p_f \langle \mathbb{D}(y,\mathbf{U}_2^{(0)})\rangle_{Y_s} \\
&= \Big(\sum_{i,j=1}^{3} \langle \mathbb{D}(y,\mathbf{U}_2^{(ij)})\rangle_{Y_s} \otimes \mathbb{J}^{(ij)}\Big) : \mathbb{D}(x,\precapprox_s) + \frac{m}{\lambda_0} p_f\, \langle \mathbb{D}(y,\mathbf{U}_2^{(0)})\rangle_{Y_s},
\end{aligned}$$

$$\begin{aligned}
&\lambda_0\, \mathfrak{N}^{(0)} : \big((1-m)\mathbb{D}(x,\precapprox_s) + \langle \mathbb{D}(y,\mathbf{W}_s)\rangle_{Y_s}\big) - m p_f \mathbb{I} \\
&= \lambda_0\, \mathfrak{N}^{s} : \mathbb{D}(x,\precapprox_s) - p_f \mathbb{C}^s,
\end{aligned}$$

$$\begin{aligned}
\langle \nabla_y \cdot \mathbf{W}_s\rangle_{Y_s} &= \sum_{i,j=1}^{3} \langle \nabla_y \cdot \mathbf{U}_2^{(ij)}\rangle_{Y_s} D_{ij} + \frac{m}{\lambda_0} p_f \langle \nabla_y \cdot \mathbf{U}_2^{(0)}\rangle_{Y_s} \\
&= \big(\sum_{i,j=1}^{3} \langle \nabla_y \cdot \mathbf{U}_2^{(ij)}\rangle_{Y_s} \mathbb{J}^{ij}\big) : \mathbb{D}(x,\precapprox_s) + \big(\frac{m}{\lambda_0} \langle \nabla_y \cdot \mathbf{U}_2^{(0)}\rangle_{Y_s}\big) p_f,
\end{aligned}$$

where

$$\mathfrak{N}^s = \mathfrak{N}^{(0)} : \Big((1-m)\sum_{i,j=1}^{3} \mathbb{J}^{ij} \otimes \mathbb{J}^{ij} + \sum_{i,j=1}^{3} \langle \mathbb{D}(y,\mathbf{U}_2^{(ij)})\rangle_{Y_s} \otimes \mathbb{J}^{(ij)}\Big), \tag{5.26}$$

$$\mathbb{C}^s = m\mathbb{I} - \langle \mathbb{D}(y,\mathbf{U}_2^{(0)})\rangle_{Y_s}, \tag{5.27}$$

$$\mathbb{C}_0^s = \sum_{i,j=1}^{3} \langle \nabla_y \cdot \mathbf{U}_2^{(ij)}\rangle_{Y_s} \mathbb{J}^{ij},\ \ c_0^s = \langle \nabla_y \cdot \mathbf{U}_2^{(0)}\rangle_{Y_s}. \tag{5.28}$$

The derivation of the equation for $\mathbf{W}^{(f)}$ repeats in its main features the arguments of the previous section. We choose the test functions φ^ε in (5.3) as

$$\varphi^\varepsilon(\mathbf{x},t) = h(\mathbf{x},t)\, \varphi_0(\frac{\mathbf{x}}{\varepsilon}),$$

where h is a smooth function with a compact support in Ω and $\varphi_0(\mathbf{y})$ is a smooth 1-periodic solenoidal function with a compact support in Y_f. After the limit as $\varepsilon \to 0$ and localisation we arrive at the differential equation

$$\varrho_f \frac{\partial^2 \mathbf{W}^{(f)}}{\partial t^2} = \mu_1 \nabla \cdot \mathbb{D}\big(y, \frac{\partial \mathbf{W}^{(f)}}{\partial t}\big) - \nabla_y \Pi^{(f)} - \nabla p_f \tag{5.29}$$

in the domain Y_f for $t > 0$. Here, as in previous section, we also must define a 1-periodic in $\mathbf{y}$ function $\Pi^{(f)}(\mathbf{x},t,\mathbf{y})$, which appears due to the choice of test functions.

The missing equation is derived from the continuity equation in its integral form (5.5) in the same way as in the previous section and coincides with (4.19).

According to [8] and [9] the system (5.29), (4.19) supplies with the boundary condition

$$\mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y}) = \mathbf{w}_s(\mathbf{x},t) \tag{5.30}$$

on the boundary γ, and the homogeneous initial conditions

$$\mathbf{W}^{(f)}(\mathbf{x},0,\mathbf{y}) = \frac{\partial \mathbf{W}^{(f)}}{\partial t}(\mathbf{x},0,\mathbf{y}) = 0. \tag{5.31}$$

Problem (4.19), (5.29)-(5.31) has been solved in [9]:

$$\begin{aligned}
\mathbf{W}^{(f)} &= \gtrapprox_s(\mathbf{x},t) + \sum_{i=1}^{3}\int_0^t \mathbf{W}_i^{(f)}(\mathbf{y},t-\tau)\big(\frac{\partial p_f}{\partial x_i}(\mathbf{x},\tau) + \varrho_f \frac{\partial^2 w_{s,i}}{\partial \tau^2}(\mathbf{x},\tau)\big)d\tau \\
&= \gtrapprox_s(\mathbf{x},t) + \sum_{i=1}^{3}\int_0^t \big(\mathbf{W}_i^{(f)}(\mathbf{y},t-\tau)\otimes \mathbf{e}_i\big)\cdot\big(\nabla p_f(\mathbf{x},\tau) + \varrho_f \frac{\partial^2 \mathbf{w}_s}{\partial \tau^2}(\mathbf{x},\tau)\big)d\tau,
\end{aligned}$$

$$\Pi^{(f)}(\mathbf{x},t,\mathbf{y}) = \sum_{i=1}^{3}\int_0^t \Pi_i^{(f)}(\mathbf{y},t-\tau)\big(\frac{\partial p_f}{\partial x_i}(\mathbf{x},\tau) + \varrho_f \frac{\partial^2 w_{s,i}}{\partial \tau^2}(\mathbf{x},\tau)\big)d\tau,$$

where $\gtrapprox_s = (w_{s,1}, w_{s,2}, w_{s,3})$ and $\{\mathbf{W}_i^{(f)}, \Pi_i^{(f)}\}$, $i = 1,2,3$, are solutions to the following periodic initial boundary value problem

$$\varrho_f \frac{\partial^2 \mathbf{W}_i^{(f)}}{\partial t^2} = \mu_1 \nabla\cdot\mathbb{D}\big(y, \frac{\partial \mathbf{W}_i^{(f)}}{\partial t}\big) - \nabla_y \Pi_i^{(f)}, \quad (\mathbf{y},t)\in Y_f\times(0,T), \tag{5.32}$$

$$\nabla_y\cdot \mathbf{W}_i^{(f)}(\mathbf{y},t) = 0,\ (\mathbf{y},t)\in Y_f\times(0,T), \tag{5.33}$$

$$\mathbf{W}_i^{(f)}(\mathbf{y},0) = 0,\ \ \varrho_f \frac{\partial \mathbf{W}_i^{(f)}}{\partial t}(\mathbf{y},0) = -\mathbf{e}_i, \quad \mathbf{y}\in Y_f, \tag{5.34}$$

$$\mathbf{W}_i^{(f)}(\mathbf{y},t) = 0,\ (\mathbf{y},t)\in\gamma\times(0,T). \tag{5.35}$$

Thus,

$$\begin{aligned}
\mathbf{w}^{(f)}(\mathbf{x},t) &= \int_{Y_f} \mathbf{W}^{(f)}(\mathbf{x},t,\mathbf{y})dy \\
&= m \gtrapprox_s(\mathbf{x},t) + \int_0^t \mathbb{B}_3^{(f)}(t-\tau)\cdot\big(\nabla p(\mathbf{x},\tau) + \varrho_f \frac{\partial^2 \mathbf{w}_s}{\partial \tau^2}(\mathbf{x},\tau)\big)d\tau,
\end{aligned} \tag{5.36}$$

where

$$\mathbb{B}_3^{(f)}(t) = \sum_{i=1}^{3}\int_{Y_f} \mathbf{W}_i^{(f)}(\mathbf{y},t)dy\otimes \mathbf{e}_i. \tag{5.37}$$

Differential equations (5.9)-(5.11), (5.20), and (5.36), boundary conditions (5.14)-(5.17), initial conditions (5.18) and state equations (5.19) and (5.26)-(5.28) constitute the mathematical model (III) of seismics in composite media.

6. One dimensional model for the case (I): numerical implementations

Direct problem. For the sake of simplicity we consider the space, which consists of the following subdomains: $\Omega_1 = \{x\in\mathbb{R} : 0 < x < H_1\}$, $\Omega_2 = \{x\in\mathbb{R} : H_1 < x <$

$H_2\}$, and $\Omega_3 = \{x \in \mathbb{R} : x > H_2\}$. Differential equations (3.8), (3.9), (3.12), and (3.13) result in

$$\frac{1}{\hat{c}^2(x)}\frac{\partial^2 p}{\partial t^2} = \operatorname{div}\big(\frac{1}{\hat{\rho}(x)}\nabla(p + m\frac{\nu_0}{\hat{c}_f^2}\frac{\partial p}{\partial t})\big)$$

where

$$\frac{1}{\hat{c}^2} = \frac{m}{\hat{c}_f^2} + \frac{(1-m)}{\hat{c}_s^2}$$

and $\hat{\rho} = m\rho_f + (1-m)\rho_s$ are respectively average wave propagation velocity and average density of the medium.

Applying now the Fourier transformation we arrive at

$$\frac{d^2\hat{P}}{dX^2} + \frac{\hat{\rho}\omega^2}{(1 - \frac{m\nu_0}{c_f^2}i\omega)\hat{c}^2}\hat{P} = 0 \tag{6.1}$$

where $\hat{P}(x,\omega)$-the pressure obtained after Fourier transform.

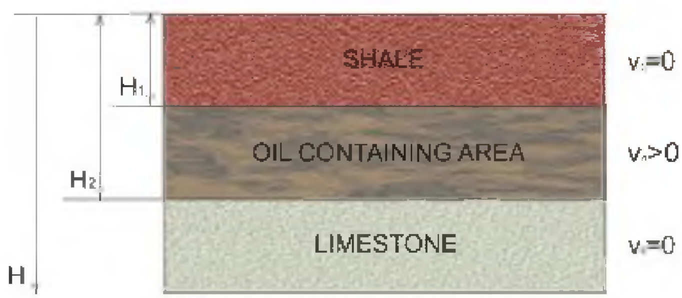

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FIGURE 3. Scheme of arrangement of layers

Depending on the exact physical properties, the sedimentary rock zone is divided into three subdomains. The value of the geometry of pores, viscosity of fluid, density of rock, and velocity of seismic wave considered in each layers to be different. In the experiment in order to get numerical solution, it's assumed that the first medium is a shale, the second medium is oil saturated sandstone, and the third medium is a limestone (see Fig.3).

Let us suppose that there is a plane wave which propagates from ∞. Then the general solution of equation (6.1) for $-\infty < X \le H_1$ in the case $\nu_0 = 0$ is written down as:

$$\hat{P}_1 = \exp\Big\{\frac{i\omega\sqrt{\hat{\rho}_1}}{\hat{c}_1}x\Big\} + A_2\exp\Big\{\frac{-i\omega\sqrt{\hat{\rho}_1}}{\hat{c}_1}x\Big\}. \tag{6.2}$$

The general solution of equation (6.1) for $H_1 \le x < H_2$ in the case $\nu_0 > 0$ is represented as:

$$\hat{P}_2 = B_1\exp\Big\{\frac{i\omega\sqrt{\hat{\rho}_2}}{\hat{c}_2\sqrt{1 - \frac{m\nu_0}{c_f^2}i\omega}}x\Big\} + B_2\exp\Big\{\frac{-i\omega\sqrt{\hat{\rho}_2}}{\hat{c}_2\sqrt{1 - \frac{m\nu_0}{c_f^2}i\omega}}x\Big\}. \tag{6.3}$$

Finally the general solution for $x \ge H_2$ in the case $\nu_0 = 0$ will be the following:

$$\hat{P}_3 = D_1\exp\Big\{i\omega\frac{\sqrt{\hat{\rho}_3}}{\hat{c}_3}x\Big\}. \tag{6.4}$$

Continuity condition in contact media will be:

$$[\hat{P}_1 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_1]_{H_{1-0}} = [\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{1+0}} \tag{6.5}$$

$$\hat{c}_1^2\frac{d}{dx}[\hat{P}_1 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_1]_{H_{1-0}} = \hat{c}_2^2\frac{d}{dx}[\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{1+0}} \tag{6.6}$$

$$[\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{2-0}} = [\hat{P}_3 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_3]_{H_{2+0}} \tag{6.7}$$

$$\hat{c}_2^2\frac{d}{dx}[\hat{P}_2 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_2]_{H_{2-0}} = \hat{c}_3^2\frac{d}{dx}[\hat{P}_3 - i\omega\frac{m\nu_0}{c_f^2}\hat{P}_3]_{H_{2+0}} \tag{6.8}$$

These relations are nothing else but the system of linear algebraic equations for the coefficients A_2, B_1, B_2, D_1 which can be easily resolved by any direct method. These coefficients are used in order to construct the solution in time frequency domain and after inverse Fourier transform in time the solution in the time domain can be easily recovered (see Fig.4).

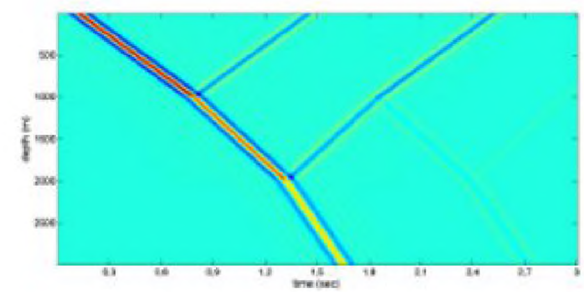

FIGURE 4. Propagation of seismic waves in different layers

Inverse problem. In inverse problem [15] except $\hat{P}(x,\omega)$ the values H_1, H_2, $\hat{c}_2$, ν_0, m are unknown as well. To determine these values one needs some additional information about solution of the direct problem - data of inverse problem. Usually they are given as function $\overline{P}(\omega)$ at $X = 0$. The most widespread way is to search for these values by minimization of the data misfit functional being L_2 norm of the difference of given functions and computed for some current values of unknown parameters:

$$F_i(H_1^i, H_2^i) = \int_{\omega_1}^{\omega_n} |\hat{P}_i(\omega, H_1^i, H_2^i) - \overline{P}(\omega, H_1, H_2)|^2 d\omega \to 0 \tag{6.9}$$

$$F_i(H_1^i, \tilde{c}_2^i) = \int_{\omega_1}^{\omega_n} |\hat{P}_i(\omega, H_1^i, \tilde{c}_2^i) - \overline{P}(\omega, H_1, \hat{c}_2)|^2 d\omega \to 0 \tag{6.10}$$

$$F_i(H_2^i, \tilde{c}_2^i) = \int_{\omega_1}^{\omega_n} |\hat{P}_i(\omega, H_2^i, \tilde{c}_2^i) - \overline{P}(\omega, H_2, \hat{c}_2)|^2 d\omega \to 0 \tag{6.11}$$

Here $\overline{P}(\omega,\dots,\dots)$ is the given wave fields at $X = 0$, while $\hat{P}(\omega,\dots,\dots)$ are wave fields computed for some current values of the desired parameters.,

In our numerical experiments the minimum is searched by the Nelder-Mead technique ([17], Fig.5).

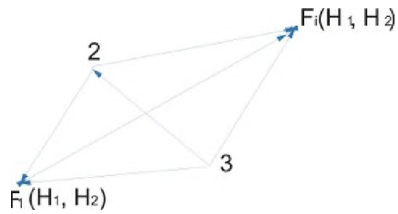

FIGURE 5. Simple scheme of Nelder-Mead for two variables regular simplex

Recovery of H_1 *and* H_2. Behavior of the data misfit functional for this statement is represented in Figures 6 and 7. As one can see this functional is convex and has the unique minimum point. Therefore this inverse problem is well resolved.

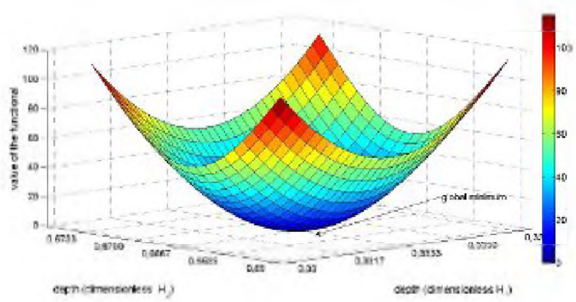

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FIGURE 6. Minimization of the functional $F(H_1, H_2)$.

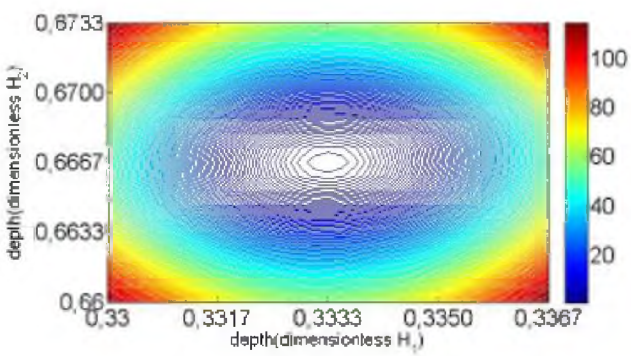

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FIGURE 7. Level line of the functional $F(H_1, H_2)$.

Recovery of H_1 *and* c_2. Now we come to the non convex functional and therefore inverse problem may have few solutions (see Figures 8 and 9).

Recovery of H_2 *and* c_2. This statement also generates non convex functional, but now it has excellent resolution with respect to H_2 (see Figures 10 and 11).

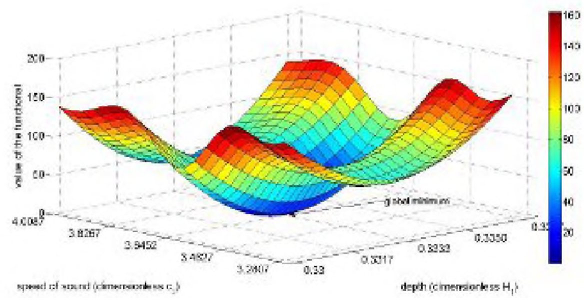

FIGURE 8. Minimization of the functional $F(H_1, \hat{c}_2)$.

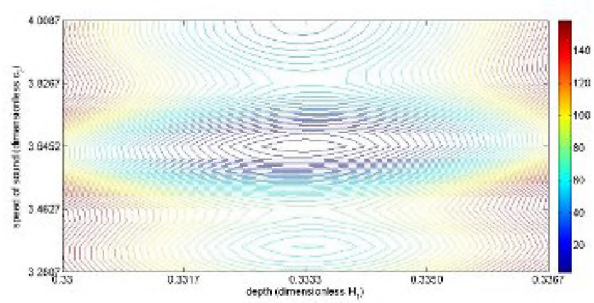

FIGURE 9. Level line functional $F(H_1, \hat{c}_2)$.

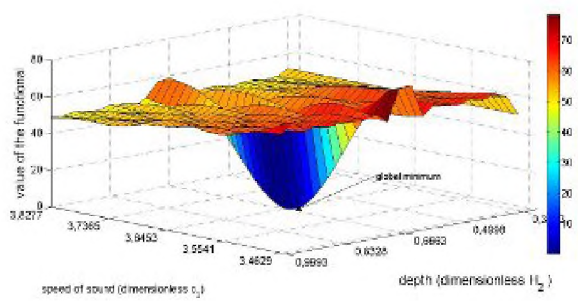

FIGURE 10. Minimization of the functional $F(H_2, \hat{c}_2)$.

Conclusions. In this publication we have shown how to derive mathematical models for composite media using its microstructure. As a rule, there is some set of models depending on given criteria μ_0, $\lambda_0, \dots$ of the physical process in consideration. For a fixed set of criteria the corresponding model describes some of the main features of the process.

In the paper the simplest inverse problem was dealt with - recovery of elastic parameters of the layer by Nelder-Mead algorithm. In the future we are planning to establish connection upscaling procedure and scattered waves and apply on this

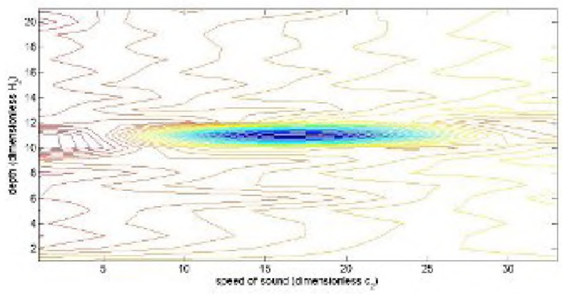

FIGURE 11. Level line functional $F(H_2, \hat{c}_2)$.

base recent developments of true-amplitude imaging on the base of Gaussian beams for both reflected and scattered waves [18, 19].

REFERENCES

[1] M. Biot; *Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range.* Journal of the Acoustical Society of America. **28**, 168–178 (1955).

[2] M. Biot; *Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher frequency range.* Journal of the Acoustical Society of America. **28**, 179–191 (1955).

[3] M. Biot; *Generalized theory of seismic propagation in porous dissipative media.* J. Acoust. Soc. Am. **34**, 1256–1264 (1962).

[4] J. G. Berryman; *Seismic wave attenuation in fluid-saturated porous media*, J. Pure Appl. Geophys. (PAGEOPH) 128, 423-432 (1988).

[5] J. G. Berryman; *Effective medium theories for multicomponent poroelastic composites*, ASCE Journal of Engineering Mechanics 132 (5), 519–531 (2006).

[6] R. Burridge, J. B. Keller; *Poroelasticity equations derived from microstructure*, Journal of Acoustic Society of America 70, No. 4 (1981), 1140 – 1146.

[7] E. Sanchez-Palencia; *Non-Homogeneous Media and Vibration Theory, Lecture Notes in Physics*, Vol. 129, Springer-Verlag, 1980.

[8] A. Meirmanov; *Mathematical models for poroelastic flows*, Atlantis Press, Paris, 2013.

[9] A. Meirmanov; *A description of seismic seismic wave propagation in porous media via homogenization.* SIAM J. Math. Anal. **40**, N3, 1272–1289 (2008).

[10] A. Mikelić; *On the Efective Boundary Conditions Between Diferent Flow Regions. Proceedings of the 1. Conference on Applied Mathematics and Computation Dubrovnik*, Croatia, September 1318 (1999), 21-37.

[11] A. Bensoussan, J. L. Lions, G. Papanicolau; Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.

[12] N. Bakhvalov, G. Panasenko; *Homogenization: averaging processes in periodic media*, Math. Appl., Vol. 36, Kluwer Academic Publishers, Dordrecht, 1990.

[13] G. Nguetseng; *A general convergence result for a functional related to the theory of homogenization.* SIAM J. Math. Anal. 20 (1989), 608–623.

[14] G. Nguetseng; *Asymptotic analysis for a stiff variational problem arising in mechanics*, SIAM J. Math. Anal. 21 (1990), 1394–1414.

[15] A. Fichtner; *Full Seismic Waveform Modelling and Inversion, Springer- Verlag Berlin Heidelberg.* (2011) pp. 113–140.

[16] M. I. Protasov, G. V. Reshetova, V. A. Tcheverda; *Fracture detection by Gaussian beam imaging of seismic data and image spectrum analysis*, Geophysical Prospecting, in-press (2015).

[17] J. H. Mathews, K. D. Fink; *Numerical methods using Matlab.* 4th edition, 2004, Prentice-Hall Inc., chapter 8, pp.430–436.

[18] M. I. Protasov, V. A. Tcheverda; *True amplitude imaging by inverse generalized Radon transform based on Gaussian beam decomposition of the acoustic Green's function*, Geophysical Prospecting, **59**(2), 197–209 (2011).

[19] M. I. Protasov, V. A. Tcheverda; *True-amplitude elastic Gaussian beam imaging of multicomponent walk-away VSP data*, Geophysical Prospecting, **60**(6), 1030–1042 (2012).

Addendum posted on November 28, 2016

The editor from Zentralblatt informed us that a big portion of this article coincides with the article

"Seismic in composite media: elastic and poroelastic components" by Anvarbek Meirmanov; Saltanbek Talapedenovich Mukhambetzhanov and Marat Nurtas (Sib. Elekron. Mat. Izv. 13, 75-88) (2016) (Zbl 06607056).

The Electron. J. Differential Equations requested an explanation from the authors. They replied that two co-authors submitted the manuscript to two different journals, and each eventually published it without consulting the other. They write,

> It's my fault that I did not control the process. There is not any other explanation. Now I do not know what I should do. Maybe the best way here is to remove the paper from the site, if it is possible.
> I apologize once again,
> yours sincerely,
> Anvarbek Meirmanov.

Since the article is already published, the EJDE editor posted this explanation. We recommend that co-authors inform each other about their submissions. End of addendum.

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