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SHORT COMMUNICATIONS

On Multidimensional Difference Operators and Equations

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Abstract— We study the solvability of multidimensional difference equations in Sobolev–Slobodetskii spaces. In the simplest model case, we describe the solvability picture for such equations. In the general case, we present conditions for the Fredholm property and a theorem on the zero index.

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1. The main goal of the present paper is to show that a key role in the theory of (multidimensional) difference equations is played not by the order of an equation (or an operator) but by some integer topological characteristic, which determines the number of additional (boundary) conditions or conditions to be imposed on the right-hand side of the equation so as to ensure the unique solvability in certain function spaces. We use the theory of the classical Riemann boundary value problem and one-dimensional singular integral equations as well as the general ideas of the theory of boundary value problems for elliptic pseudodifferential equations [1–5].

2. A multidimensional difference equation is an equation of the form

$$(Au)(x) \equiv \sum_{|k|=0}^{\infty} a_k(x)u(x+\alpha_k) = v(x), \qquad x \in D, \tag{1}$$

where $D \subset \mathbb{R}^m$ is a convex set, k is a multi-index, $|k| = k_1 + \cdots + k_m$, and $\{\alpha_k\} \subset D$.

Difference equations of the form (1) arise in numerous theoretical and applied studies. The classical theory of difference equations with constant coefficients can be found in the monographs [6, p. 543; 7, p. 384]; however, studies on general boundary value problems in nonsmooth domains (in contrast to the smooth case [5, p. 200]) also necessitate considering difference equations with variable coefficients [8, 9].

We denote the compactification of the space $\mathbb{R}^m$ by $\dot{\mathbb{R}}^m$. In the present paper, we consider the case in which $D = \mathbb{R}^m_+ \equiv \{x \in \mathbb{R}^m : \ x = (x_1, \dots, x_m),\, x_m > 0\}$.

The *symbol* of the operator A is defined as the multiple series

$$\sigma_A(x,\xi) = \sum_{|k|=0}^{\infty} a_k(x) e^{-i\alpha_k\cdot\xi}, \qquad x \in D, \qquad \xi \in \mathbb{R}^m,$$

provided that $\sigma_A(x,\xi) \in C(\overline{D} \times \dot{\mathbb{R}}^m)$.

The symbol $\sigma_A(x,\xi)$ is said to be *elliptic* if $\sigma_A(x,\xi) \neq 0$, $(x,\xi) \in \overline{D} \times \dot{\mathbb{R}}^m$. By

$$\widetilde{u}(\xi) \equiv (Fu)(\xi) \equiv \int\limits_{\mathbb{R}^m} e^{-ix\cdot\xi} u(x)\, dx$$

we denote the Fourier transform of a function u. The Sobolev–Slobodetskii space $H^s(\mathbb{R}^m)$, $s \in \mathbb{R}$, consists of (generalized) functions with finite norm

$$\|u\|_s = \left(\int\limits_{\mathbb{R}^m} |\widetilde{u}(\xi)|^2 (1+|\xi|)^{2s}\, d\xi \right)^{1/2}.$$

3. The case of constant coefficients in Eq. (1) can be studied in detail by the Wiener–Hopf method. Set

$$(Bu)(x) \equiv \sum_{|k|=0}^{\infty} a_k u(x+\alpha_k) = v(x), \qquad x \in \mathbb{R}^m_+. \tag{2}$$

A *factorization* of the elliptic symbol $\sigma_B(\xi)$ is a representation of the form

$$\sigma_B(\xi) = \sigma_+(\xi)\sigma_-(\xi),$$

where the factors $\sigma_\pm(\xi) \in L_\infty(\mathbb{R}^m)$ admit analytic continuation into the upper and lower half-plane, respectively, in the last variable ξ_m for given $\xi' = (\xi_1, \ldots, \xi_{m-1})$.

The *factorization index* of the elliptic symbol $\sigma_B(\xi)$ is the number $\varkappa \in \mathbb{Z}$ defined as the increment divided by 2π of the argument of $\sigma_B(\xi', \xi_m)$ as ξ_m varies from $-\infty$ to $+\infty$. By virtue of homotopic properties of the index, one can readily show that it is independent of ξ'. It can be represented by the Stieltjes integral

$$\varkappa = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} d\arg\sigma_B(\cdot, \xi_m).$$

We introduce the following notation: $H_{\xi'}$ is the Hilbert transform with respect to the variable ξ_m; i.e.,

$$(H_{\xi'}\widetilde{u})(\xi', \xi_m) = \frac{1}{\pi i}\text{v.p.} \int\limits_{-\infty}^{+\infty} \frac{\widetilde{u}(\xi', \eta_m)\, d\eta_m}{\xi_m - \eta_m};$$

$P_{\xi'} = (I + H_{\xi'})/2$, and $\sigma_\pm(\xi)$ are the elements of factorization of the symbol

$$(\xi_m + i|\xi'| + i)^{\varkappa}(\xi_m - i|\xi'| - i)^{-\varkappa}\sigma_B(\xi);$$

P' is the operator of integration with respect to the last variable; i.e.,

$$(P'\widetilde{u})(\xi') = \int\limits_{-\infty}^{+\infty} \widetilde{u}(\xi', \xi_m)\, d\xi_m;$$

$H^s(D)$ is the space of (generalized) functions in $H^s(\mathbb{R}^m)$ supported in $\overline{D}$; $H^s_0(D)$ is the space of generalized functions consisting of $v \in S'(D)$ [linear functionals on the Schwarz space $S(\mathbb{R}^m)$ of infinitely differentiable functions rapidly decreasing at infinity and supported in $\overline{D}$] that admit continuation lv into the entire space $H^s(\mathbb{R}^m)$; $\widetilde{h}(\xi) = (\xi_m - i|\xi'| - i)^{-\varkappa}\sigma_-^{-1}(\xi)\widetilde{lv}(\xi)$; $\widetilde{g} = P_{\xi'}(Q_n^{-1}\widetilde{h})$, and $Q_n(\xi)$ is an arbitrary polynomial of degree n without real zeros.

Theorem 1. *If* $|s| < 1/2$ *and* $\varkappa = 0$, *then Eq.* (2) *has a unique solution* $u \in H^s(\mathbb{R}^m_+)$ *for an arbitrary right-hand side* $v \in H^s_0(\mathbb{R}^m_+)$. *If* $s + \varkappa < -1/2$ *and* $n \in \mathbb{N}$ *satisfies the condition* $-1/2 < s + \varkappa + n < 0$, *then* $\dim\operatorname{Ker} B = n$, *and all solutions of Eq.* (2) *in the space* $H^s(\mathbb{R}^m_+)$ *are defined by the formulas*

$$\tilde{u}(\xi', \xi_m) = (\xi_m + i|\xi'| + i)^{\varkappa}\sigma_+^{-1}(\xi)Q_n(\xi)\widetilde{g}(\xi', \xi_m) + (\xi_m + i|\xi'| + i)^{\varkappa}\sigma_+^{-1}(\xi', \xi_m)\sum_{k=1}^{n}\widetilde{c}_k(\xi')\xi_m^{k-1},$$

where $c_k \in H^{s_k}(\mathbb{R}^{m-1})$ *are arbitrary functions and* $s_k = -\varkappa + k - 1/2$, $k = 1, \dots, n$. *If* $s + \varkappa > 1/2$ *and* $n \in \mathbb{N}$ *satisfies the condition* $0 < s + \varkappa - n < 1/2$, *then* $\dim \operatorname{Coker} B = n$ *and the solution of Eq.* (2) *admits the representation*

$$\tilde{u}(\xi) = \sum_{k=1}^{n} \frac{\widetilde{c}_k(\xi')}{\sigma_+(\xi)\Lambda^{k-\varkappa}(\xi', \xi_m)} + \frac{1}{\sigma_+(\xi)\Lambda^{n-\varkappa}(\xi', \xi_m)}(P_{\xi'}\Lambda^n \widetilde{h})(\xi', \xi_m),$$

where $\widetilde{c}_k = (P'\Lambda^{k-1})\widetilde{h}$, $c_k \in H^{s_k}(\mathbb{R}^{m-1})$, $s_k = s + \varkappa - k + 1/2$, *and* $\Lambda(\xi', \xi_m) = \xi_m + |\xi'| + i$.

4. The local principle [10, p. 23] allows one to make some conclusions for equations with variable coefficients.

Theorem 2. *Let* $\sigma_A(x,\xi) \in C(\overline{D} \times \dot{\mathbb{R}}^m)$ *and* $\sigma_A(x,\xi) \neq 0$ *for all* $(x,\xi) \in \overline{D} \times \dot{\mathbb{R}}^m$, *and let the following relation hold*:

$$\int_{-\infty}^{+\infty} d \arg \sigma(\cdot, \cdot, \xi_m) = 0.$$

Then A *is a Fredholm operator of index zero in the space* $H^s(\mathbb{R}^m_+)$, $|s| < 1/2$.

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