The Periodic Cauchy Kernel, the Periodic Bochner Kernel, and Discrete Pseudo-Differential Operators

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Abstract. We introduce a discrete pseudo-differential operator in an appropriate discrete functional space and study the invertibility properties for such simplest operators in certain canonical domains of an Euclidean space. We construct special projectors for studying these operators according to a type of canonical domain and show how these operators are related to special boundary value problems for holomorphic functions of several variables.

INTRODUCTION

A classical pseudo-differential operator in the Euclidean space $\mathbb{R}^m$ is defined by ([1, 2])

$$(Au)(x) = \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} \widetilde{A}(x,\xi)e^{i(\xi-y)}\tilde{u}(\xi)d\xi dy,$$

where the sign ~ over a function denotes its Fourier transform

$$\widetilde{u}(\xi) = \int\limits_{\mathbb{R}^m} u(x)e^{ix\cdot\xi}dx.$$

Given a function u_d of a discrete variable $\tilde{x} \in \mathbb{Z}^m$, we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \widetilde{u}_d(\xi) = \sum_{\tilde{x}\in\mathbb{Z}^m} e^{i\tilde{x}\cdot\xi}u(\tilde{x}), \quad \xi \in \mathbb{T}^m,$$

where the partial sums are taken over cubes

$$Q_N = \{\tilde{x} \in \mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \max_{1\le k\le m} |\tilde{x}_k| \le N\}.$$

Let $D \subset \mathbb{R}^m$ be a cone, $D_d \equiv D \cap \mathbb{Z}^m$, and $L_2(D_d)$ be a space of functions of a discrete variable defined on D_d, and $A(\tilde{x})$ be a given function of a discrete variable $\tilde{x} \in \mathbb{Z}^m$. We consider the operators

$$(A_d u_d)(\tilde{x}) = \int_{\mathbb{T}^m} \sum_{\tilde{y}\in D_d} e^{i(\tilde{y}-\tilde{x})\cdot\xi}\widetilde{A}(\xi)\tilde{u}_d(\xi)d\xi, \quad \tilde{x} \in D_d, \tag{1}$$

and introduce the function

$$\widetilde{A}_d(\xi) = \sum_{\tilde{x}\in\mathbb{Z}^m} e^{i\tilde{x}\cdot\xi}A(\tilde{x}), \quad \xi \in \mathbb{T}^m.$$

Definition 1. *The function $\widetilde{A}_d(\xi)$ is called a symbol of the operator A_d, and this symbol is called an elliptic symbol if $\tilde{A}_d(\xi) \neq 0, \forall\xi \in \mathbb{T}^m$.*

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DISCRETE PSEUDO-DIFFERENTIAL OPERATORS

Definition 2. *The formula (1) defines a discrete pseudo-differential operator in the canonical domain D_d.*

Example 1. If $K(x), x \in \mathbb{R}^m \setminus \{0\}$, is a Calderon–Zygmund kernel, then the corresponding operator is defined by [3]

$$(K_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in \mathbb{Z}^m, \tilde{y} \neq \tilde{x}} K(\tilde{x} - \tilde{y}), \quad \tilde{x} \in \mathbb{Z}^m.$$

Example 2. If a first order finite difference of a discrete variable $\tilde{x}_k$ is defined by

$$\delta_k u_d(\tilde{x}) = u_d(\tilde{x}_k + 1) - u_d(\tilde{x}_k),$$

then the discrete Laplacian is

$$(\Delta_d u_d)(\tilde{x}) = \sum_{k=1}^{m} \left(u_d(\tilde{x}_k + 2) - 2u_d(\tilde{x}_k + 1) + u_d(\tilde{x}_k)\right),$$

and its symbol is the function

$$\sigma_{\Delta_d}(\xi) = \sum_{k=1}^{m} (e^{i\xi_k} - 1)^2.$$

Discrete Sobolev–Slobodetskii Spaces

Let $H^s(\mathbb{Z}^m)$ denotes the space of functions of a discrete variable for which

$$||u_d||_s^2 \equiv \int_{\mathbb{T}^m} |\widetilde{u}_d(\xi)|^2 (1 + |\sigma_{\Delta_d}(\xi)|)^s d\xi < +\infty.$$

We say a discrete operator A_d has order α if its symbol $\widetilde{A}_d(\xi)$ satisfies the condition

$$|\widetilde{A}_d(\xi)| \sim (1 + |\sigma_{\Delta_d}(\xi)|)^{\frac{\alpha}{2}}.$$

The class of such symbols will be denoted by $S_\alpha(\mathbb{T}^m)$.

Lemma 1. *Pseudo-differential operator A_d of order α is a linear bounded operator $H^s(\mathbb{Z}^m) \to H^{s-\alpha}(\mathbb{Z}^m)$.*

Indeed,

$$||A_d u_d||_s^2 = \int_{\mathbb{T}^m} |\widetilde{A}_d(\xi)\widetilde{u}_d(\xi)|^2 (1 + |\sigma_{\Delta_d}(\xi)|)^s d\xi \leq c \int_{\mathbb{T}^m} |\widetilde{u}_d(\xi)|^2 (1 + |\sigma_{\Delta_d}(\xi)|)^{s+\alpha} d\xi.$$

DISCRETE PROJECTORS AND COMPLEX VARIABLES

Let us denote by P_{D_d} the projection operator on D_d, $P_{D_d} : L_2(\mathbb{Z}^m) \to L_2(D_d)$, so that for an arbitrary function $u_d \in L_2(\mathbb{Z}^m)$, $(P_{D_d} u_d)(\tilde{x}) = u_d(\tilde{x})$ if $\tilde{x} \in D_d$, and $u_d(\tilde{x}) = 0$ otherwise.

Half-Space Case and Periodic Cauchy kernel

If we consider a half-space case, then the Fourier image of the operator P_{D_d} can be evaluated ([3, 4]) and we'll demonstrate it in the following

Example 3. *If $D = \mathbb{R}^m_+$, then*

$$(F_d P_{D_d} u_d)(\xi', \xi_m) = \frac{1}{4\pi i} \lim_{\tau \to 0+} \int_{-\pi}^{\pi} u_d(\xi', \eta_m) \cot \frac{\xi_m - \eta_m + i\tau}{2} d\eta_m.$$

Thus we use a periodic one-dimensional Riemann problem with a parameter $\xi' \in \mathbb{T}^{m-1}$ which is the following. Finding a pair of functions $\Phi^{\pm}(\xi', \xi_m)$ which are boundary values of holomorphic functions in half-strips $\Pi_{\pm} = \{z \in \mathbb{C} : z = \xi_m \pm i\tau, \tau > 0\}$ such that these are satisfied a linear relation

$$\Phi^{+}(\xi)(\xi', \xi_m) = G(\xi', \xi_m)\Phi^{-}(\xi)(\xi', \xi_m) + g(\xi), \quad \xi \in \mathbb{T}^m,$$

for almost all $\xi' \in \mathbb{T}^{m-1}$, where $G(\xi), g(\xi)$ are given periodic functions. It looks like the classical cases [5, 6].

Conical Case and Periodic Bochner kernel

Let D be a sharp convex cone, and $\overset{*}{D}$ be a conjugate cone for D, i.e.,

$$\overset{*}{D} = \{x \in \mathbb{R}^m : x \cdot y > 0, \ y \in D\}.$$

Let $T(\overset{*}{D}) \subset \mathbb{C}^m$ be a set of the type $\mathbb{T}^m + i\,\overset{*}{D}$. For $\mathbb{T}^m \equiv \mathbb{R}^m$ such a domain of multidimensional complex space is called a radial tube domain over the cone $\overset{*}{D}$ ([7, 8, 9]). We introduce the function

$$B_d(z) = \sum_{\tilde{x} \in D_d} e^{i\tilde{x} \cdot z}, \quad z = \xi + i\tau, \quad \xi \in \mathbb{T}^m, \quad \tau \in \overset{*}{D},$$

and define the operator

$$(B_d u)(\xi) = \lim_{\tau \to 0} \int_{\mathbb{T}^m} B_d(z - \eta) u_d(\eta) d\eta.$$

Lemma 2. *For arbitrary $u_d \in L_2(\mathbb{Z}^m)$, the following property*

$$F_d P_{D_d} u_d = B_d F_d u_d$$

holds.

Let us define the subspace $A(\mathbb{T}^m) \subset L_2(\mathbb{T}^m)$ consisting of functions which admit a holomorphic continuation into $T(\overset{*}{D})$ and satisfy the condition

$$\sup_{\tau \in \overset{*}{D}} \int_{\mathbb{T}^m} |\tilde{u}_d(\xi + i\tau)|^2 d\xi < +\infty.$$

In other words, the space $A(\mathbb{T}^m) \subset L_2(\mathbb{T}^m)$ consists of boundary values of holomorphic in $T(\overset{*}{D})$ functions.

Let us denote

$$B(\mathbb{T}^m) = L_2(\mathbb{T}^m) \ominus A(\mathbb{T}^m),$$

so that $B(\mathbb{T}^m)$ is a direct complement of $A(\mathbb{T}^m)$ in $L_2(\mathbb{T}^m)$.

A jump problem

We formulate the problem in the following way: finding a pair of functions $\Phi^{\pm}, \Phi^{+} \in A(\mathbb{T}^m), \Phi^{-} \in B(\mathbb{T}^m)$, such that

$$\Phi^{+}(\xi) - \Phi^{-}(\xi) = g(\xi), \quad \xi \in \mathbb{T}^m, \tag{2}$$

where $g(\xi) \in L_2(\mathbb{T}^m)$ is given.

Lemma 3. *The operator $B_d : L_2(\mathbb{T}^m) \to A(\mathbb{T}^m)$ is a bounded projector. A function $u_d \in L_2(D_d)$ iff its Fourier transform $\tilde{u}_d \in A(\mathbb{T}^m)$.*

Theorem 1. *The jump problem has unique solution for arbitrary right-hand side from $L_2(\mathbb{T}^m)$.*

Example 4. *If $m = 2$ and D is the first quadrant in a plane then a solution of a jump problem is given by formulas*

$$\Phi^{+}(\xi) = \frac{1}{(4\pi i)^2} \lim_{\tau \to 0} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cot \frac{\xi_1 + i\tau_1 - t_1}{2} \cot \frac{\xi_2 + i\tau_2 - t_2}{2} g(t_1, t_2) dt_1 dt_2$$

$$\Phi^{-}(\xi) = \Phi^{+}(\xi) - g(\xi), \quad \tau = (\tau_1, \tau_2) \in D.$$

A general statement

It looks as follows. Finding a pair of functions $\Phi^{\pm}, \Phi^{+} \in A(\mathbb{T}^m), \Phi^{-} \in B(\mathbb{T}^m)$, such that

$$\Phi^{+}(\xi) = G(\xi)\Phi^{-}(\xi) + g(\xi), \quad \xi \in \mathbb{T}^m, \tag{3}$$

where $G(\xi), g(\xi)$ are given periodic functions. If $G(\xi) \equiv 1$, we have the jump problem (2).

Like classical studies [5, 6], we want to use a special representation for an elliptic symbol to solve the problem (3).

Periodic wave factorization

Let us denote $H^s(D_d)$ a subspace of $H^s(\mathbb{Z}^m)$ consisting of functions of discrete variable $\tilde{x}$ for which their supports belong to $\overline{D_d}$, and $\widetilde{H}^s(D_d), \widetilde{H}^s(\mathbb{Z}^m)$ their Fourier images.

Lemma 4. *For $|s| < 1/2$, the operator B_d is a bounded projector $\widetilde{H}^s(\mathbb{Z}^m) \to \widetilde{H}^s(D_d)$, and a jump problem has unique solution $\Phi^{+} \in \widetilde{H}^s(D_d), \Phi^{-} \in \widetilde{H}^s(\mathbb{Z}^m \setminus D_d)$ for arbitrary $g \in \widetilde{H}^s(\mathbb{Z}^m)$.*

Definition 3. Periodic wave factorization for elliptic symbol $\tilde{A}(\xi)$ is called its representation in the form

$$\tilde{A}_d(\xi) = \tilde{A}_{\neq}(\xi)\tilde{A}_{=}(\xi)$$

where the factors $A^{\pm 1}_{\neq}(\xi), A^{\pm 1}_{=}(\xi)$ admit bounded holomorphic continuation into domains $T(\pm \overset{*}{D})$.

Theorem 2. *If $|s| < 1/2$ and the elliptic symbol $\tilde{A}_d(\xi) \in S_\alpha(\mathbb{T}^m)$ admits periodic wave factorization, then the operator A_d is invertible in the space $H^s(D_d)$.*

CONCLUSION

These considerations have to be useful for statements of boundary value problems for discrete elliptic pseudo-differential equations in canonical non-smooth domains. Such boundary value problems will appear when an index of the wave factorization is not zero. Moreover we hope to establish a correspondence between discrete and continual [9] cases and to describe a limit transfer from discrete case to continual one.

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