On Some New Classes of Pseudo-Differential Operators

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Abstract. We study Fredholm properties for a special class of elliptic pseudo-differential operators. Using a local principle we give boundedness theorems for such operators and describe their Fredholm properties in Sobolev–Slobodetskii spaces of a variable order. For a half-space case we introduce a certain operator family which helps describing Fredholm properties.

INTRODUCTION

Pseudo-differential operators and equations were born in past century (see, for example, [1–3]) and now this theory lives and develops. In this paper we would like to enlarge class of spaces of spaces in which such pseudo-differential operators can act, and describe their Fredholm properties in these spaces. We use spaces of variable order and consider also pseudo-differential operators of variable order. First similar constructions were introduced in [4] (and further there were some generalizations and refinements), but our considerations contain new results related to boundedness and Fredholm properties and based on a local principle. Moreover we study a family of spaces of a variable order and intend to apply this methodology to appearing boundary value problems. Also we hope these consideration will be useful for more complicated situations when we deal with a cone instead of a half-space [5].

Local Sobolev–Slobodetskii Spaces

Let $s : \mathbb{R}^m \to \mathbb{R}$ be an arbitrary function satisfying the following conditions:

- the finite limit $\lim\limits_{|x|\to+\infty} s(x)$ exists,
- the function $s(x)$ satisfies the Lipschitz condition in $\mathbb{R}^m$, i. e. there is a positive constant $C > 0$ such that

$$|s(x_1) - s(x_2)| \le c|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^m.$$

For fixed $x \in \mathbb{R}^m$ we introduce the following definition.

Definition 1 *By definition a local Sobolev–Slobodetskii space $H^{s(x)}(\mathbb{R}^m)$ consists of distributions with finite value*

$$||u||_{s(x)} \equiv \left(\int\limits_{\mathbb{R}^m} (1 + |\xi|)^{2s(x)} |\tilde{u}(\xi)|^2 d\xi\right)^{1/2},$$

where $\tilde{u}$ denotes the Fourier transform of the function u.

The value $||u||_{s(x)}$ is called a local H^s-norm of the function u.

For a brevity we will write $H^{s(x)}$ instead of $H^{s(x)}(\mathbb{R}^m)$, and if we speak on $H^{s(x)}$-functions with supports in a certain domain $D \subset \mathbb{R}^m$ then we write $H^{s(x)}(D)$.

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Definition 2 *The universal Sobolev–Slobodetskii super-space H^{s_M} is called the space which includes all local Sobolev–Slobodetskii spaces with finite norm*

$$||u||_{s_M} = \sup_{x\in\mathbb{R}^m} ||u||_{s(x)}.$$

The universal Sobolev–Slobodetskii subspace H^{s_m} is called the space which is inside of all local Sobolev–Slobodetskii spaces with finite norm

$$||u||_{s_m} = \inf_{x\in\mathbb{R}^m} ||u||_{s(x)}.$$

Obviously,

$$||u||_{s_m} \leq ||u||_{s(x)} \leq ||u||_{s_M}, \quad \forall x \in \mathbb{R}^m.$$

Let us denote $S(\mathbb{R}^m)$ the Schwartz class of infinitely differentiable rapidly decreasing at infinity functions; this class is dense in each local Sobolev–Slobodetskii space [3].

Lemma 1 *For $u \in S(\mathbb{R}^m)$ we have estimates*

$$|(1 + |\xi|)^{s_1} - (1 + |\xi|)^{s_2}| \leq c_1|s_1 - s_2| \cdot (1 + |\xi|)^l,$$

$$|||u||_{s(x_1)} - ||u||_{s(x_2)}| \leq c_2|x_1 - x_2| \cdot ||u||_{s_M}$$

for a certain $l \in [s_1, s_2]$.

Let H_x be a family of Hilbert spaces parametrized by points $x \in \mathbb{R}^m$. We denote $||u||_x$ the norm of element $u \in H^{s(x)}$ and assume that S is a dense subset in $H^{s(x)}, \forall x \in \mathbb{R}^m$. For $u \in S$ we define the functional

$$f(x, u) = ||u||_x.$$

Definition 3 *We say that a family of Hilbert spaces $\{H_x\}_{x\in\mathbb{R}^m}$ is a local continuous family in the point $x_0 \in \mathbb{R}^m$ if for fixed $u \in S$ the functional $f(x, u)$ is continuous in the point x_0.*

Lemma 2 *The family $H^{s(x)}$ is a local continuous family.*

OPERATORS OF A VARIABLE ORDER

Definition 4 *Given function $A(x, \xi)$ defined in $\mathbb{R}^m \times \mathbb{R}^m$ a pseudo-differential operator A is called an operator of the following type*

$$(Au)(x) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} A(x, \xi) e^{i(x-y)\xi} u(y)\, dy d\xi, \quad x \in \mathbb{R}^m. \tag{1}$$

The function $A(x, \xi)$ is called a symbol of the operator A.

Let $\alpha : \mathbb{R}^m \to \mathbb{R}$ be a function with the same properties as $s(x)$.

Definition 5 *The class $E_{\alpha(x)}$ consists of functions $A(x, \xi)$ defined in $\mathbb{R}^m \times \mathbb{R}^m$ and satisfying the conditions*

$$c_1(1 + |\xi|)^{\alpha(x)} \leq |A(x, \xi)| \leq c_2(1 + |\xi|)^{\alpha(x)} \tag{2}$$

and for each point $x_0 \in \mathbb{R}^m$ there exists a neighborhood U_{x_0} such that for all $x \in U_{x_0}$ the following inequality

$$|A(x, \xi) - A(x_0, \xi)| \leq c_3|x - x_0|(1 + |\xi|)^{\alpha(x)} \tag{3}$$

holds, where c_1, c_2, c_3 are positive constants. The function $\alpha : \mathbb{R}^m \to \mathbb{R}$ is called a variable order of a pseudo-differential operator (symbol).

If x_0 is an infinity then we need to require satisfying the following inequality

$$|A(x, \xi) - A(\infty, \xi)| \leq c_4|x|^{-1}(1 + |\xi|)^{\alpha(x)}, \tag{3'}$$

instead of the inequality (3).

Example. A very simple example is so called fractional Laplacian of a variable order. Its symbol is

$$A(x, \xi) = (1 + \xi_1^2 + \xi_2^2 + \cdots + \xi_m^2)^{\alpha(x)}. \tag{4}$$

All conditions mentioned above are satisfied.

Boundedness Theorems

If we fix $x_0 \in \mathbb{R}^m$ then the operator A_{x_0} is an operator with the symbol $A(x_0, \xi)$.

Lemma 3 *If $A(x_0, \xi) \in E_{\alpha(x_0)}$ then the operator A_{x_0} is bounded in local Sobolev–Slobodetskii space, $A_{x_0} : H^{s(x_0)} \to H^{s(x_0)-\alpha(x_0)}$.*

This property is called a local boundedness.

Definition 6 *An operator A with the symbol $A(x, \xi)$ is called a local bounded operator if for each point $x_0 \in \mathbb{R}^m$ the operator $A_{x_0} : H^{s(x_0)} \to H^{s(x_0)-\alpha(x_0)}$ with the symbol $A(x_0, \xi)$ is bounded.*

Theorem 4 *If $A(x, \xi) \in E_{\alpha(x)}$ then the function $x_0 \longmapsto \|A_{x_0}\|_{s(x_0)}$ is continuous at each point $x_0 \in \mathbb{R}^m$ including the infinity.*

Remark 1 *It means the property of a local boundedness is conserved in enough small neighborhood.*

Theorem 5 *If an operator A with the symbol $A(x, \xi)$ is local bounded then it is bounded $A : H^{s_M} \to H^{(s-\alpha)_m}$.*

Remark 2 *It seems it is enough to require a local boundedness on a certain countable set including the infinity.*

Compact Operators in Spaces $H^{s(x)}$

Let $\{T_x : H^{s_1(x)} \to H^{s_2(x)}\}_{x\in\mathbb{R}^m}$ be a family of linear bounded operators.

Lemma 6 *If the linear bounded operator $T_{x_0} : H^{s_1(x_0)} \to H^{s_2(x_0)}$ is compact then there is a neighborhood U_{x_0} of the point x_0 such that for all $x_1 \in U_{x_0}$ the operator $T_{x_1} : H^{s_1(x_1)} \to H^{s_2(x_1)}$ is compact.*

Let $\psi_{x_0}(x) \in C_0^\infty(\mathbb{R}^m)$ be a function equals to 1 in some neighborhood of the point x_0.

Corollary 7 *An operator with the symbol $\psi_{x_0}(x)A(x, \xi) - A(x_0, \xi)$ is compact in some neighborhood of the point x_0 as operator $H^{s(x)} \to H^{s(x)-\alpha(x)}$.*

Fredholmness

Definition 7 *An operator A_{x_0} with the symbol $A(x_0, \xi)$ ($x_0 \in \mathbb{R}^m$ is fixed) we call the local representative of the operator A in the point x_0.*

Theorem 8 *If local representatives $A_{x_0} : H^{s(x_0)} \to H^{s(x_0)-\alpha(x_0)}$ of the operator A are invertible at each point of $\mathbb{R}^m$ including the infinity then the operator $A : H^{s_M} \to H^{(s-\alpha)_m}$ has a Fredholm property.*

Remark 3 *It is enough to require an invertibility for the local representatives in a certain dense set .*

A HALF-SPACE CASE

Definition 8 *Let $A(x, \xi)$ be a function defined in $\mathbb{R}^m_+ \times \mathbb{R}^m$. A pseudo-differential operatorA in a half-space with the symbol $A(x, \xi)$ is called an operator of the following type*

$$(Au)(x) = \int_{\mathbb{R}^m_+} \left(\int_{\mathbb{R}^m} A(x, \xi) e^{i(x-y)\xi} u(y)\, d\xi \right) dy, \quad x \in \mathbb{R}^m_+. \tag{5}$$

We consider the following equation in a half-space

$$(Au)(x) = v(x), \quad x \in \mathbb{R}^m_+, \tag{6}$$

related to such an operator.

According to [3] we introduce the space $H^{s(x)}(\mathbb{R}^m_+)$ of functions from $H^{s(x)}(\mathbb{R}^m)$ with support in $\overline{\mathbb{R}^m_+}$, this is space of solutions, and the space $H^{s(x)}_0(\mathbb{R}^m_+)$ of functions (distributions) from $S'(\mathbb{R}^m)$ with support in $\overline{\mathbb{R}^m_+}$, these must admit analytical continuation in a whole $H^{s(x)}(\mathbb{R}^m)$ with finite norm

$$||v||^+_{s(x)} = \inf ||\ell v||_{s(x)},$$

where *infimum* is taken over all continuations ℓ.

For studying a Fredholmness of the equation (6) we use a local principle. We extract two types of local representatives.

1) For inner point $x_0 \in \mathbb{R}^m_+$ this is well-known operator

$$(A_{x_0}u)(x) = \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} A(x_0,\xi)e^{i(x-y)\xi}u(y)\,dyd\xi, \quad x \in \mathbb{R}^m.$$

2) For boundary point $x'_0 = (x'_0, 0) \in \mathbb{R}^{m-1}$ (we use notation $x_0 = (x'_0, x^{(0)}_m)$ for the $x_0 \in \mathbb{R}^m$) it will be the operator

$$(A_{x'_0}u)(x) \longmapsto \int\limits_{\mathbb{R}^m_+}\left(\int\limits_{\mathbb{R}^m} A(x'_0,\xi)e^{i(x-y)\xi}u(y)\,d\xi\right)dy, \quad x \in \mathbb{R}^m_+. \tag{7}$$

Thus, we have two operator families.

Theorem 9 *If all operators of the family $A_{x'_0} : H^{s(x')}(\mathbb{R}^m_+) \to H^{s(x')-\alpha(x')}(\mathbb{R}^m_+)$ are bounded then the operator (5) is bounded as operator $H^{s_M}(\mathbb{R}^m_+) \to H^{(s-\alpha)_m}(\mathbb{R}^m_+)$.*

If all operators of two operator family are invertible then the operator (5) is Fredholm operator as operator $H^{s_M}(\mathbb{R}^m_+) \to H^{(s-\alpha)_m}(\mathbb{R}^m_+)$.

Variable Index of Factorization

This is key point for studying pseudo-differential equations in a half-space. We will give a corresponding definition and show its applicability to the equation (7) with fixed $x' \in \mathbb{R}^{m-1}$.

Definition 9 *Factorization of the symbol $A(x',\xi)$ on a variable ξ_m is called its representation in the form*

$$A(x',\xi',\xi_m) = A_+(x',\xi',\xi_m)A_-(x',\xi',\xi_m),$$

where the factors $A_+(A_-)$ admit analytical continuation on ξ_m in upper (lower) complex half-plane $\xi_m \pm i\tau, \tau > 0$, under almost all $\xi' \in \mathbb{R}^{m-1}$ and satisfy estimates

$$|A_+(x',\xi',\xi_m)| \le c_1(1 + |\xi'| + |\xi_m| + |\tau|)^{æ(x')},$$

$$|A_-(x',\xi',\xi_m)| \le c_1(1 + |\xi'| + |\xi_m| + |\tau|)^{\alpha(x')-æ(x')}, \quad \forall\tau \in \mathbb{R}.$$

The function $æ(x')$, $x' \in \mathbb{R}^{m-1}$, is called a variable index of factorization.

Let us note that such factorization can be constructed effectively with a help of the Cauchy type integral [3].

Below we will assume that the function $æ(x')$ has the same properties as $s(x')$.

A Local Solvability and Boundary Conditions

Now there is a question how one can use the theorem 9. In other words we need conditions which guarantee an invertibility for local representatives of boundary operator.

Theorem 10 *Let*

$$|æ(x') - s(x')| < 1/2, \quad \forall x' \in \mathbb{R}^{m-1}. \tag{8}$$

Then the operator (7) is invertible as an operator $H^{s(x')}(\mathbb{R}^m_+) \to H^{s(x')-\alpha(x')}(\mathbb{R}^m_+)$ under each fixed x'.

Collecting the theorems 9, 10 we obtain a Fredholm property for pseudo-differential operator (equation) in a half-space.

Theorem 11 *Let* $A(x,\xi) \in E_{\alpha(x)}$. *If*

$$|æ(x') - s(x')| < 1/2, \quad \forall x' \in \mathbb{R}^{m-1}, \quad \forall x' \in R^{m-1},$$

then the operator (5) has a Fredholm property as an operator $H^{s_M}(\mathbb{R}^m_+) \to H^{(s-\alpha)_m}(\mathbb{R}^m_+)$.

There are situations for which the condition (8) does not hold. We consider here one of possible variants.

Let $æ(x') - s(x') = m + \delta, |\delta| < 1/2, m \in \mathbb{N}, \forall x' \in \mathbb{R}^{m-1}$. Now for fixed x' even the operator (7) is non-invertible, but we know a general solution of corresponding equation [3]. We will briefly describe its form to explain appearing boundary operators.

Thus, we consider an equation with the operator (7) and right-hand side $f \in H_0^{s(x'_0)-\alpha(x'_0)}(\mathbb{R}^m_+)$.

We will remind that for fixed $x'_0 \in \mathbb{R}^{m-1}$ a general solution of the equation

$$(A_{x'_0}u)(x) = f(x), \quad x \in \mathbb{R}^m_+$$

with the operator (7) in Fourier image has the following form [3]

$$\tilde{u}(\xi) = A_+^{-1}(x'_0,\xi)P_m(\xi)\Pi_+P_m^{-1}(\xi)A_-^{-1}(x'_0,\xi)\widetilde{\ell f}(\xi) + A_+^{-1}(x'_0,\xi)\sum_{k=1}^{m}\tilde{c}_k(x'_0,\xi')\xi_m^{k-1}, \tag{9}$$

where the operator Π_+ is the Cauchy type integral

$$(\Pi_+\tilde{u})(\xi) = \frac{i}{2\pi}\lim_{\tau\to 0+}\int_{-\infty}^{+\infty}\frac{\tilde{u}(\xi'.\eta_m)\,d\eta_m}{\xi_m - \eta_m + i\tau}dd\eta_m,$$

$P_m(\xi) \in E_m$ is an arbitrary polynomial, $c_k(x'_0,\xi')$ are arbitrary functions from $H^{s_k(x'_0)}(\mathbb{R}^{m-1})$, $s_k(x'_0) = s(x'_0) - æ(x'_0) + k - 1/2, k = 1,\cdots,m$.

To determine uniquely c_k we choose m bounded pseudo-differential operators $B_j : H^{s(x'_0)}(\mathbb{R}^m_+) \to H^{s(x'_0)-\alpha_j(x'_0)}(\mathbb{R}^m_+)$ with symbols $B_j(x_0,\xi), j = 1,\cdots,m$, and γ is a restriction operator on the hyperplane $x_m = 0$ so that operators γB_j are bounded $H^{s(x'_0)}(\mathbb{R}^m_+) \to H^{s(x'_0)-\alpha_j(x'_0)-1/2}(\mathbb{R}^{m-1})$ under the condition $s(x'_0) - \alpha_j(x'_0) - 1/2 > 0, \forall x'_0 \in \mathbb{R}^{m-1}$. Let us introduce a new local sobolev–Slobodetskii space as a direct sum

$$H^{s_\alpha(x'_0)} \equiv H^{s(x'_0)}(\mathbb{R}^m_+) \oplus \sum_{j=1}^{m} H^{s(x'_0)-\alpha_j(x'_0)-1/2}(\mathbb{R}^{m-1})$$

For the point $x'_0 \in \mathbb{R}^{m-1}$ we introduce a local operator $H^{s(x'_0)}(\mathbb{R}^m_+) \to H^{s_\alpha(x'_0)}$ by the formula

$$B_{x'_0}u = (A_{x'_0}u, \gamma B_1 u, \gamma B_1 u \cdots, \gamma B_m u). \tag{10}$$

Therefore, the operator $B_{x'_0}$ is given by pasting m additional operators together with a family of additional spaces to the operator $A_{x'_0}$ at each point $x'_0 \in \mathbb{R}^{m-1}$.

We can conclude as above that under our assumptions all such operators are locally bounded, and it implies a boundedness of the (10).

One can study an invertibility of the local operator (10) by the Fourier transform, it permits to reduce an identification problem for the functions c_k to unique solvability of some $m \times m$-system of linear algebraic equations. Non-vanishing for a determinant of the latter system is necessary and sufficient condition for an invertibility of the local operator $B_{x'_0}$.

Remark 4 *The left case* $æ(x') - s(x') = -m + \delta, |\delta| < 1/2, m \in \mathbb{N}, \forall x' \in \mathbb{R}^{m-1}$, *can dt considered analogously. For this case we extend the domain of* $A_{x'}$, *and we past to the domain additional local Sobolev–Slobodetskii spaces with additional unknowns in these spaces.*

CONCLUSION

It seems a lot of results obtained by a local principle can be transferred on operators and equations of a variable order. In our opinion more interesting case is studying such operators and equations on manifolds with non-smooth boundaries like [5–9].

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