*Electronic Journal of Differential Equations*, Vol. 2014 (2014), No. 227, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# A COMPACTNESS LEMMA OF AUBIN TYPE AND ITS APPLICATION TO DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. Let $\Omega \subset \mathbb{R}^n$ be a regular domain and $\Phi(s) \in C_{\rm loc}(\mathbb{R})$ be a given function. If $\mathfrak{M} \subset L_2(0,T;W_2^1(\Omega)) \cap L_\infty(\Omega \times (0,T))$ is bounded and the set $\{\partial_t \Phi(v) | \, v \in \mathfrak{M}\}$ is bounded in $L_2(0,T;W_2^{-1}(\Omega))$, then there is a sequence $\{v_k\} \in \mathfrak{M}$ such that $v_k \rightharpoonup v \in L^2(0,T;W_2^1(\Omega))$, and $v_k \to v$, $\Phi(v_k) \to \Phi(v)$ a.e. in $\Omega_T = \Omega \times (0,T)$. This assertion is applied to prove solvability of the one-dimensional initial and boundary-value problem for a degenerate parabolic equation arising in the Buckley-Leverett model of two-phase filtration. We prove existence and uniqueness of a weak solution, establish the property of finite speed of propagation and construct a self-similar solution.

## 1. Introduction

In the present work, we establish an Aubin-type compactness lemma [3, 9] with a nonlinear restriction and then apply it to solving a nonlinear degenerate parabolic equation, which arises from a special case of the one-dimensional Buckley-Leverett model of two-phase filtration [2, 4]. By now, there exist numerous compactness results of this type, see, e.g., [5] for a review of the available literature; however none of them seems to be applicable to the problem which we address in this note. The mathematical model consists of two Darcy's systems of filtration

$$\mathbf{v}^{(1)} = -\frac{k^{(1)}(s)}{\mu^{(1)}} \nabla p^{(1)}, \quad \frac{\partial s}{\partial t} + \nabla \cdot \mathbf{v}^{(1)} = 0,$$
$$\mathbf{v}^{(2)} = -\frac{k^{(2)}(s)}{\mu^{(2)}} \nabla p^{(2)}, \quad -\frac{\partial s}{\partial t} + \nabla \cdot \mathbf{v}^{(2)} = 0$$

for two immiscible fluids with velocities $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$, pressures $p^{(1)}$ and $p^{(2)}$, and viscosities $\mu^{(1)}$ and $\mu^{(2)}$. The unknown concentration $s$ of the first fluid is defined from the state equation

$$p^{(1)} - p^{(2)} = P_c(s),$$

where $k^{(1)}(s)$, $k^{(2)}(s)$, and $P_c(s)$ are given functions. For example,

$$k^{(1)}(s) = s, \quad k^{(2)}(s) = 1 - s, \quad P_c(s) = s. \tag{1.1}$$

2000 *Mathematics Subject Classification.* 35B27, 46E35, 76R99.
*Key words and phrases.* Compactness lemma; two-phase filtration; nonlinear PDE; degenerate parabolic equations.

Submitted September 25, 2104. Published October 27, 2014.

To simplify the problem, let us assume that in addition to (1.1),

$$\mu^{(1)} = \mu^{(2)} = 1.$$

Under this assumption, the one-dimensional system transforms into

$$v^{(1)} = -s\frac{\partial p^{(1)}}{\partial x}, \quad \frac{\partial s}{\partial t} + \frac{\partial v^{(1)}}{\partial x} = 0, \tag{1.2}$$

$$v^{(2)} = -(1-s)\frac{\partial p^{(2)}}{\partial x}, \quad -\frac{\partial s}{\partial t} + \frac{\partial v^{(2)}}{\partial x} = 0, \tag{1.3}$$

$$\frac{\partial p^{(1)}}{\partial x} - \frac{\partial p^{(2)}}{\partial x} = \frac{\partial s}{\partial x}. \tag{1.4}$$

Equations (1.2)-(1.4) lead to the relations

$$v = v^{(1)} + v^{(2)} \equiv v(t), \tag{1.5}$$

$$v^{(1)} = -\frac{\partial u}{\partial x} + v(t)\, s, \quad v^{(2)} = \frac{\partial u}{\partial x} + v(t)\,(1-s), \tag{1.6}$$

in which

$$u = \Psi(s) = \int_0^s \xi\,(1-\xi)d\xi, \quad \frac{\partial u}{\partial x} = s(1-s)\frac{\partial s}{\partial x}.$$

Gathering these relations we arrive at the differential equation

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} - v(t)\, s\Big) \tag{1.7}$$

for the concentration $s$ of the first fluid.

Let us consider the problem of displacement of the second fluid by the first one in the domain $\Omega = (-1, 1)$. The physical meaning of the problem imposes the following boundary and initial conditions:

$$v^{(1)}(-1,t) = v_0(t), \quad v^{(2)}(-1,t) = 0, \quad t > 0, \tag{1.8}$$

$$s(x,0) = s_0(x) \in [0,1], \quad -1 < x < 1. \tag{1.9}$$

By (1.5) and the boundary condition (1.9),

$$v(t) = v_0(t),$$

$$\frac{\partial u}{\partial x}(-1,t) + v_0(t)\ (1 - s(-1,t)) = 0.$$

In accordance with the general theory of second-order PDE, one has to impose one more boundary condition for the concentration $s(x,t)$ on the boundary $\{x = 1\}$. To complete the mathematical formulation of the problem, we set

$$s(1,t) = 0, \quad t > 0, \tag{1.10}$$

which means that the concentration of the first fluid equals zero on the right end-point of the interval $(-1, 1)$. Let us notice that although condition (1.10) has a clear physical meaning, it can be substituted by other physically reasonable conditions. Our choice of (1.10) is explained by the simplicity of the resulting mathematical problem, as well as by the fact that thus far none of the other possible boundary conditions on the line $x = 1$ has been given a due justification.

We prove that problem (1.7)-(1.10) has a unique weak solution $s(x,t)$, and that this solution possesses the property of finite speed of propagation of disturbances from the data. A solution of problem (1.7)-(1.10) is constructed as the limit of a sequence of solutions of regularized nondegenerate problems.

The property of finite speed of propagation is intrinsic for solutions of nonlinear degenerate parabolic equations and is not displayed by the solutions of any linear equation. For the solutions of equation (1.7) this property is described as follows: if $s(x,0)=0$ in some interval $(x_0-R, x_0+R) \subset (-1,1)$, then there are functions $r^{\pm}(t) > 0$ such that $s(x,t)=0$ in the interval $(x_0 - r^-(t), x_0 + r^+(t))$ for all sufficiently small $t$. Since (1.7) generates also at the level $s=1$, the same is true for the set $\{s=1\}$.

An exhaustive analysis of the property of finite speed of propagation, as well as a detailed review of the bibliography, can be found in [6, 7, 1]. The approach of [6, 7] is based on comparison of solutions of a degenerate one-dimensional PDE with a family of travelling wave solutions, the method developed in [1] relies on the analysis of ordinary differential inequalities for "local energies" associated with the solutions of a PDE under study. Both methods are applicable to problem (1.7)-(1.10). In the general case when $0 \le s_0(x) \le 1$ the property of finite speed of propagation is proved by means of the local energy method. In the special case when

$$s_0(x) = 1 \text{ in } (-1,0) \text{ and } s_0(x) = 0 \text{ in } (0,1) \tag{1.11}$$

this property immediately follows from the existence of a self-similar solution. Let the initial data satisfy (1.11). Then problem (1.7)-(1.10) can be written as the boundary-value problem for the second-order ordinary differential equation for the function $\bar{w}(\xi) = s(x,t)$, which depends on the variable

$$\xi = \frac{x}{\sqrt{t}} - \frac{1}{\sqrt{t}} \int_0^t v_0(\tau)\,d\tau.$$

The problem for $\bar{w}(\xi)$ has the form

$$\Psi''(\bar{w}) + \frac{\xi}{2}\bar{w}' = 0, \quad 0 < \bar{w} < 1 \quad \text{for } -\xi_* < \xi < \xi_*, \tag{1.12}$$

$$\bar{w}(-\xi_*) = 1, \quad \bar{w}(\xi_*) = 0, \quad \Psi'(\bar{w})(-\xi_*) = \Psi'(\bar{w})(\xi_*) = 0 \tag{1.13}$$

with a finite $\xi^* > 0$ to be defined. Uniqueness of weak solution of problem (1.7)-(1.10) means that $s(x,t) \equiv \bar{w}(\xi)$ for $t \in (0, t_*)$, where $t_* = \min\{t^-,\, t^+\}$ with

$$\int_0^{t^-} v_0(\tau)\,d\tau - \xi_* \sqrt{t^-} = -1, \quad \int_0^{t^+} v_0(\tau)\,d\tau + \xi_* \sqrt{t^+} = 1.$$

The curves

$$x = R^{\pm}(t) \equiv \int_0^t v_0(\tau)\,d\tau \pm \xi_* \sqrt{t}$$

demarcate the domains where $s(x,t) \equiv 1$, $s(x,t) \in (0,1)$, or $s(x,t) \equiv 0$. We failed to find any result regarding solvability of problem (1.12)-(1.13) and provide the proof of existence and uniqueness of self-similar solution in the concluding section of this work.

Throughout the text we use the traditional notation in [8, 9] for the functional spaces and norms.

## 2. Main results

**Lemma 2.1.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, $\Omega_T = \Omega \times (0,T)$, and let $\Phi \in C_{\rm loc}(\mathbb{R})$ be a given function. Denote by $\mathfrak{M}$ a bounded set in $L_2(0,T; W_2^1(\Omega)) \cap L_\infty(\Omega_T)$.*

*Assume that for every* $v \in \mathfrak{M}$ *and every* $\varphi \in L_2(0,T;\overset{\circ}{W}{}_2^1(\Omega))$ *the function* $s(x,t) = \Phi\left(v(x,t)\right)$ *satisfies the inequality*

$$\Big|\int_0^T \int_\Omega \frac{\partial s}{\partial t}\,\varphi\,dx\,dt\Big|^2 \leqslant M\int_0^T\int_\Omega |\nabla\varphi|^2\,dx\,dt \tag{2.1}$$

*with an independent of* $v \in \mathfrak{M}$ *constant* $M$*. Then there exists a sequence* $\{v_m\} \subset \mathfrak{M}$*, which converges weakly in* $L_2(0,T;W_2^1(\Omega))$ *and almost everywhere in* $\Omega_T$*, and the corresponding sequence* $\{s_m\}$*,* $s_m = \Phi(v_m)$*, converges almost everywhere in* $\Omega_T$*.*

**Definition 2.2.** We say that the pair of measurable and bounded in $\Omega_T$ functions $s$ and $u = \Psi$ is a weak solution of problem (1.7)-(1.10) if

$$\begin{aligned}&\int_0^T\int_\Omega \Big(s\frac{\partial\varphi}{\partial t} + u\frac{\partial^2\varphi}{\partial x^2} + v_0(t)\,s\frac{\partial\varphi}{\partial x}\Big)\,dx\,dt\\ &= -\int_\Omega s_0(x)\varphi(x,0)\,dx - v_0(t)\int_0^T \varphi(-1,t)\,dt\end{aligned} \tag{2.2}$$

for every smooth function $\varphi$ satisfying the conditions

$$\varphi(x,T) = 0 \text{ for } -1 < x < 1, \quad \varphi(1,t) = \frac{\partial\varphi}{\partial x}(-1,t) = 0 \text{ for } 0 < t < T.$$

**Theorem 2.3.** *Let* $v_0(t)$ *be a measurable bounded function. Then for every* $T > 0$ *problem* (1.7)-(1.10) *has at least one weak solution.*

**Theorem 2.4.** *Under the conditions of Theorem 2.3 the solution of problem* (1.7)-(1.10) *is unique.*

**Theorem 2.5** (Finite speed of propagation)**.** *Under the conditions of Theorem 2.3 the solution of problem* (1.7)-(1.10) *possesses the property of finite speed of propagation:*

*(1) if* $x_0$ *and* $R$ *are such that* $s(x,0) = 0$ *for a.e.* $x \in (x_0 - R, x_0 + R)$*, then* $s = 0$ *a.e. in the domain*

$$\Big|x - x_0 + \int_0^t v(\theta)\,d\theta\Big| \le \Big(R^{1+\alpha} - \frac{C(1+\alpha)t^{1+\beta}}{1-\nu}(1+M)^{1-\nu}\Big)^{\frac{1}{1+\alpha}} \tag{2.3}$$

*with the exponents* $\nu = 6/7$*,* $\alpha = 4/3$*,* $\beta = 3/7$*,*

$$t < t_R^* = \sup\big\{t > 0 :\ -1 + R < x_0 + \int_0^t v(\theta)\,d\theta < 1 - R\big\}, \tag{2.4}$$

*and an independent of* $s$ *constant* $C$*;*

*(2) if* $x_0$ *and* $R$ *are such that* $s(x,0) = 1$ *for a.e.* $x \in (x_0 - R, x_0 + R) \subset (-1, 0]$*, then* $s = 1$ *a.e. in the domain defined by formulas* (2.3)-(2.4)*.*

Finally, in Section 7 we prove that in the special case when the initial function satisfies (1.11), the unique solution of problem (1.7)-(1.10) coincides (for small times) with the unique self-similar solution, defines by conditions (1.12)-(1.13).

## 3. Proof of Lemma 2.1

Boundedness of the set $\mathfrak{M}$ means that for all $v \in \mathfrak{M}$

$$|s(x,t)| + |v(x,t)| \le L \quad \text{a.e. in } \Omega_T \tag{3.1}$$

and

$$\int_0^T\int_\Omega |\nabla v(x,t)|^2\,dx\,dt \leqslant L \tag{3.2}$$

with a finite constant $L$. By (3.2) there is a set $G$ of full measure in $(0,T)$ such that for every $t\in G$,

$$\int_\Omega |\nabla v(x,t)|^2 dx \le M_0(t) < \infty. \tag{3.3}$$

Let $\mathcal{T}=\{t_1,t_2,\dots,t_k,\dots\}$ be a countable set of points dense in $G$. Using estimates (3.1)-(3.3) and the standard diagonal procedure we may choose a sequence $\{v_m\}\subset\mathfrak{M}$ and a function $v\in L_2\big((0,T);W_2^1(\Omega)\big)\cap L_\infty\big((0,T);L_\infty(\Omega)\big)$ such that

$$v_m(x,t_k)\to v(x,t_k) \text{ in } L_2(\Omega) \text{ and a.e. in } \Omega \text{ as } m\to\infty \text{ for every } t_k\in\mathcal{T}. \tag{3.4}$$

By the continuity of $\Phi$ and (3.4) $\Phi(v_m(x,t_k))\to\Phi(v(x,t_k))$ a.e. in $\Omega$ as $m\to\infty$. Since the sequence $\{(\Phi(v_m(x,t_k))-\Phi(v(x,t_k)))^2\}$ is uniformly bounded in $\Omega$ and tends to zero a.e. in $\Omega$, it follows from the Lebesgue dominated convergence theorem that

$$s_m(x,t_k)=\Phi\left(v_m(x,t_k)\right)\to s(x,t_k)=\Phi\left(v(x,t_k)\right) \text{ in } L_2(\Omega) \text{ and a.e. in } \Omega \tag{3.5}$$

as $m\to\infty$.

Let us fix an arbitrary point $t\in G$ and prove that $s_m(x,t)\rightharpoonup s(x,t)$ in $L_2(\Omega)$. Take $\varphi\in\mathring{W}_2^1(\Omega)$ and denote

$$I_{r,m}(t)=\Big|\int_\Omega \big(s_m(x,t)-s_r(x,t)\big)\varphi(x)dx\Big|.$$

For every $t_k\in\mathcal{T}$,

$$\begin{aligned} I_{r,m}(t) &\le \Big|\int_\Omega \big(s_m(x,t)-s_m(x,t_k)\big)\varphi(x)dx\Big| + \Big|\int_\Omega \big(s_m(x,t_k)-s_r(x,t_k)\big)\varphi(x)dx\Big| \\ &\quad + \Big|\int_\Omega \big(s_r(x,t_k)-s_r(x,t)\big)\varphi(x)dx\Big| \\ &= \Big|\int_t^{t_k}\int_\Omega \frac{\partial s_m}{\partial t}(x,\tau)\varphi(x)dxd\tau\Big| + \Big|\int_\Omega \big(s_m(x,t_k)-s_r(x,t_k)\big)\varphi(x)dx\Big|. \end{aligned}$$

Applying Hölder's inequality and using assumption (2.1) we continue this inequality as follows:

$$I_{r,m}(t)\le M\Big(\int_\Omega |\nabla\varphi|^2dx\Big)^{1/2}|t-t_k|^{1/2}+\Big|\int_\Omega \left(s_m(x,t_k)-s_r(x,t_k)\right)\varphi(x)dx\Big|. \tag{3.6}$$

Since $t\in G$ and $\mathcal{T}$ is dense in $G$, for every given $\varepsilon>0$ and $\varphi\in\mathring{W}_2^1(\Omega)$ we may find $t_k\in\mathcal{T}$ such that

$$M\Big(\int_\Omega |\nabla\varphi|^2dx\Big)^{1/2}|t-t_k|^{1/2}<\frac{\varepsilon}{2}.$$

For this $t_k$ we now choose $N$ such that

$$\Big|\int_\Omega \left(s_m(x,t_k)-s_r(x,t_k)\right)\varphi(x)dx\Big|<\frac{\varepsilon}{2}$$

for all $m,r>N$, which is always possible due to (3.5).

It follows that for every $\varepsilon>0$ there is $N>0$ such that $I_{r,m}(t)<\varepsilon$ for all $m$, $r>N$, whence

$$\int_\Omega \left(s_m(x,t)-s(x,t)\right)\varphi(x)\,dx\to 0 \quad \text{as } m\to\infty$$

for every $\varphi \in \overset{\circ}{W}{}_2^1(\Omega)$. The conclusion remains true for $\varphi \in L_2(\Omega)$ because $\overset{\circ}{W}{}_2^1(\Omega)$ is dense in $L_2(\Omega)$.

It remains to identify the limit of $\{v_m\}$. Consider now the sequence $\{v_m(x,t)\}$ with the same $t \in G$. By virtue of (3.3) there is a subsequence $\{v_{m_k}(x,t)\}$ such that

$$v_{m_k}(x,t) \to v(x,t) \text{ in } L_2(\Omega) \text{ and a.e. in } \Omega.$$

Since

$$s_{m_k}(x,t) \to s(x,t) \equiv \Phi(v(x,t)) \text{ in } L_2(\Omega) \text{ and a.e. in } \Omega,$$

it is necessary that $v_m(x,t) \to v(x,t)$ in $L_2(\Omega)$ and a.e. in $\Omega$. Finally, the inclusion $\{v_m\} \subset \mathfrak{M}$ yields the weak convergence $v_{m_k} \rightharpoonup v$ in $L_2(0,T;W_2^1(\Omega))$.

## 4. Proof of Theorem 2.3

Let $s_0^\varepsilon(x)$ be a family of smooth functions which converges to $s_0(x)$ a.e. in $\Omega$ as $\varepsilon \to 0$ and satisfies the conditions $s_0^\varepsilon(1) = 0$, $\varepsilon < s_0^\varepsilon(x) < 1-\varepsilon$. Then for all $\varepsilon > 0$ the problem

$$\begin{gathered} \frac{\partial s^\varepsilon}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial u^\varepsilon}{\partial x} - v(t)\, s^\varepsilon\Big), \quad (x,t) \in \Omega_T, \\ \frac{\partial u^\varepsilon}{\partial x}(0,t) + v_0(t)\ (1 - s^\varepsilon(0,t)) = 0, \quad t > 0, \\ s^\varepsilon(1,t) = \varepsilon, \quad t > 0, \\ s^\varepsilon(x,0) = s_0^\varepsilon(x), \quad -1 < x < 1, \end{gathered} \tag{4.1}$$

where

$$u^\varepsilon = \Psi(s^\varepsilon) = \int_0^{s^\varepsilon} \xi\,(1-\xi)d\xi,$$

has a unique smooth solution $u^\varepsilon = \Psi(s^\varepsilon)$. This solution satisfies the estimates

$$0 \leqslant s^\varepsilon(x,t) \leqslant 1, \quad (x,t) \in \Omega_T, \tag{4.2}$$

$$\int_0^T \int_\Omega \big|\frac{\partial v^\varepsilon}{\partial x}(x,t)\big|^2\, dx\, dt \leqslant M, \tag{4.3}$$

$$v^\varepsilon = \Phi^{-1}(s^\varepsilon) = \int_0^\varepsilon \sqrt{\xi(1-\xi)}d\xi, \quad s^\varepsilon = \Phi(v^\varepsilon), \tag{4.4}$$

$$\Big|\int_0^T \int_\Omega \frac{\partial s^\varepsilon}{\partial t}\,\varphi\ dx\, dt\Big|^2 \leqslant M \int_0^T \int_\Omega |\nabla\varphi|^2\, dx\, dt \tag{4.5}$$

for any $\varphi \in L_2((0,T);\overset{\circ}{W}{}_2^1(\Omega))$ with an independent of $\varepsilon$ constant $M$.

The existence and uniqueness of a smooth solution follows from the classical parabolic theory [8]. Estimate (4.2) is an immediate consequence of the maximum principle. The derive the energy estimate (4.3) we multiply equation (4.1) by $(s^\varepsilon - \varepsilon)$ and integrate by parts over $\Omega$. Finally, (4.5) follows from (4.1) after multiplication by $\varphi \in L_2((0,T);\overset{\circ}{W}{}_2^1(\Omega))$ and integration by parts.

By Lemma 2.1 there exist sequences of $\{s^\varepsilon\}$, $\{v^\varepsilon\}$ and $\{u^\varepsilon\}$ (for the sake of simplicity we keep the same notation) such that

$$s^\varepsilon(x,t) \to s, \quad v^\varepsilon \to v = \Phi^{-1}(s), \quad u^\varepsilon \to u = \Psi(s) \text{ a.e. in } \Omega_T \text{ as } \varepsilon \to 0.$$

The pair $\{s^\varepsilon, u^\varepsilon\}$ satisfies the integral identity

$$\begin{aligned}
&\int_0^T \int_\Omega \Big(s^\varepsilon \frac{\partial \varphi}{\partial t} + u^\varepsilon \frac{\partial^2 \varphi}{\partial x^2} + v_0(t) s^\varepsilon \frac{\partial \varphi}{\partial x}\Big)\, dx\, dt \\
&= -\int_\Omega s_0^\varepsilon(x)\varphi(x,0)dx + \Psi(\varepsilon)\int_0^T \frac{\partial \varphi}{\partial x}(1,t)dt - v_0(t)\int_0^T \varphi(-1,t)dt
\end{aligned} \tag{4.6}$$

for every smooth function $\varphi$ such that

$$\varphi(x,T) = 0 \text{ for } -1 < x < 1, \quad \varphi(1,t) = \frac{\partial \varphi}{\partial x}(-1,t) = 0 \text{ for } 0 < t < T. \tag{4.7}$$

Passing to the limit as $\varepsilon \to 0$ in identity (4.6), we arrive at (2.2).

## 5. Proof of Theorem 2.4

Let $\{s^{(1)}, u^{(1)}\}$ and $\{s^{(2)}, u^{(2)}\}$ be two weak solutions of problem (1.7)–(1.10). The difference $\{s, u\}$, $s = s^{(1)} - s^{(2)}$, $u = u^{(1)} - u^{(2)}$ satisfies the integral identity

$$\int_0^T \int_\Omega s\Big(\frac{\partial \varphi}{\partial t} + a(x,t)\frac{\partial^2 \varphi}{\partial x^2} + v_0(t)\frac{\partial \varphi}{\partial x}\Big)\, dx\, dt = 0 \tag{5.1}$$

for any smooth function $\varphi$ satisfying (4.7). The coefficient $a$ in (5.1) has the form

$$a(x,t) = \frac{u^{(1)} - u^{(2)}}{s^{(1)} - s^{(2)}} = \int_0^1 \frac{d\Psi}{d\xi}\Big((s^{(1)} - s^{(2)})\xi + s^{(2)}\Big)\, d\xi, \quad 0 \leqslant a(x,t) \leqslant 1.$$

Take an arbitrary smooth and finite in $\Omega_T$ function $f$ and consider the sequence $\{\varphi^{(\varepsilon)}\}$, $\varepsilon > 0$, where $\varphi^{(\varepsilon)}$ are solutions the equation

$$\frac{\partial \varphi^{(\varepsilon)}}{\partial t} + (a(x,t) + \varepsilon)\frac{\partial^2 \varphi^{(\varepsilon)}}{\partial x^2} + v_0(t)\frac{\partial \varphi^{(\varepsilon)}}{\partial x} = f(x,t) \tag{5.2}$$

in $\Omega_T$ satisfying the initial and boundary conditions (4.7). The existence of such functions follows from [8].

Multiplication of (5.2) by $\frac{\partial^2 \varphi^{(\varepsilon)}}{\partial x^2}$ and integration by parts over the domain $\Omega$ lead to the equality

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_\Omega \Big|\frac{\partial \varphi^{(\varepsilon)}}{\partial x}(x,t)\Big|^2 dx + \int_\Omega (a+\varepsilon)\Big|\frac{\partial^2 \varphi^{(\varepsilon)}}{\partial x^2}(x,t)\Big|^2 dx + \frac{1}{2}v_0(t)\Big|\frac{\partial \varphi^{(\varepsilon)}}{\partial x}(1,t)\Big|^2 \\
&= -\int_\Omega \frac{\partial \varphi^{(\varepsilon)}}{\partial x}(x,t)\frac{\partial f}{\partial x}(x,t)dx.
\end{aligned}$$

The above equality and the standard estimates for the solution of equation (5.2) lead to the estimate

$$\varepsilon \int_0^T \int_\Omega \Big|\frac{\partial^2 \varphi^{(\varepsilon)}}{\partial x^2}(x,t)\Big|^2 dx \leqslant M \tag{5.3}$$

for some constant $M$ independent of $\varepsilon$.

Finally, let us take $\varphi^\epsilon$ for the test-functions in (5.1). Straightforward computations lead to the equality

$$\varepsilon \int_0^T \int_\Omega s\Big(f(x,t) + \varepsilon\frac{\partial^2 \varphi^{(\varepsilon)}}{\partial x^2}(x,t)\Big)\, dx\, dt = 0.$$

Simplifying and passing to the limit as $\varepsilon \to 0$ and taking into account (5.3), we obtain the identity

$$\int_0^T \int_\Omega s\, f(x,t)\, dx\, dt = 0$$

for every smooth and finite in $\Omega_T$ function $f$. Thus, $s(x,t) = 0$ a.e. in $\Omega_T$.

## 6. Proof of Theorem 2.5

Let us rewrite equation (1.7) in the form

$$\frac{\partial s}{\partial t} = \frac{\partial}{\partial x}\Big(\frac{\partial \Psi(s)}{\partial x} - v(t)s\Big), \tag{6.1}$$

introduce the new variable

$$y = x + \int_0^t v(\theta)\, d\theta$$

and consider the function $w$ defined by the relation $w(y,t) = s(x,t)$. It is straightforward to check that

$$s_x(x,t) = w_x(y,t) = w_y(y,t)y_x = w_y, \quad s_t(x,t) = w_t(y,t) + w_y(y,t)v(t).$$

Fix a point $x_0 \in (0,1)$ and some $R > 0$ such that $u_0(x) = 0$ in $(x_0 - R, x_0 + R) \subset (0,1)$. The change of variables $x \mapsto y$ transforms the rectangle $(x_0 - R, x_0 + R) \times (0,T)$ in the plane of variables $(y,t)$ into the curvilinear domain $(x_+(t), x_-(t) \times (0,T)$ in the plane $(x,t)$ with the boundaries

$$x_\pm(t) = x_0 \pm R + \int_0^t v(\theta)\, d\theta.$$

Let us denote $t_R^* = \sup\{t \in (0,T) : x_+(t) < 1,\ x_-(t) > -1\}$. By (6.1), $w$ is a solution of the equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 \Psi(w)}{\partial y^2}, \quad (y,t) \in \mathcal{D} \equiv (x_0 - R, x_0 + R) \times (0, t_R^*).$$

By construction

$$\int_{\mathcal{D}} \Big(w\frac{\partial \psi}{\partial t} + \Psi(w)\frac{\partial^2 \psi}{\partial y^2}\Big) dydt = 0$$

for every regular test-function $\psi$ such that $\operatorname{supp}\psi \subseteq \mathcal{D}$. By (4.2), (4.3), (4.5) and because of the convergence $(s^\varepsilon)_t \rightharpoonup s_t$ in $L^2(0,T; W_2^{-1}(\Omega))$, $u_x^\varepsilon \rightharpoonup u_x$ in $L^2(\Omega_T)$, we conclude that $\Psi_t \in L^2(0,T; W_2^{-1}(\Omega))$. Let us fix some $\tau \in (0, t^*)$, $\rho \in (0,R)$ and take for the test-function $\psi = \Psi(w)\theta_k(t)\zeta_m(y)$ with

$$\theta_k(t) = \begin{cases} t/k & \text{if } t \in [0, 1/k), \\ 1 & \text{if } 1/k \le t < \tau - 1/k, \\ t - \tau & \text{if } \tau - 1/k < t \le \tau, \end{cases}$$

$$\zeta_m(y) = \begin{cases} \frac{|y - x_0| - \rho}{m} & \text{if } \rho - \frac{1}{m} < |y - x_0| < \rho, \\ 1 & \text{if } |y - x_0| < \rho - \frac{1}{m}, \\ 0 & \text{if } |y - x_0| \ge \rho \end{cases} \qquad k, m \in \mathbb{N}.$$

Integrating by parts and letting $m, k \to \infty$ we arrive at the equality

$$\begin{aligned} &\int_0^t \int_{x_0-\rho}^{x_0+\rho} w_t \int_0^w \xi(1-\xi)\, d\xi\, dtdy + \int_0^t \int_{x_0-\rho}^{x_0+\rho} (\Psi_y(w))^2\, dydt \\ &= \int_0^t \Psi(w)\Psi_y(w)\, dt\Big|_{x_0-\rho}^{x_0+\rho}. \end{aligned} \tag{6.2}$$

The first term on the right-hand side can be written in the form

$$\begin{aligned} \int_{x_0-\rho}^{x_0+\rho} \int_0^t \frac{\partial}{\partial t}\Big( \int_0^w \int_0^z \xi(1-\xi)\, d\xi dz\Big)\, dt\, dy &= \int_{x_0-\rho}^{x_0+\rho} \int_0^w \int_0^z \xi(1-\xi)\, d\xi dz dy \\ &= \int_{x_0-\rho}^{x_0+\rho} \frac{w^3}{6}\left(1-\frac{w}{2}\right)\, dy. \end{aligned}$$

A straightforward calculation leads to the inequality: for $w \in [0,1]$

$$\begin{aligned} \Psi(w) &= \int_0^w \xi(1-\xi)\, d\xi = \frac{w^2}{2} - \frac{w^3}{3} = \frac{w^2}{2}\big(1-\frac{2w}{3}\big) \\ &= \frac{1}{2}\Big(w^3\big(1-\frac{2w}{3}\big)^{\frac{3}{2}}\Big)^{2/3} \\ &\le \frac{1}{2}\Big(w^3\big(1-\frac{2w}{3}\big)\Big)^{2/3} \\ &\le \frac{1}{2}\Big(w^3\big(1-\frac{w}{2}\big)\Big)^{2/3} = \frac{6^{2/3}}{2}\Big(\frac{w^3}{6}\big(1-\frac{w}{2}\big)\Big)^{2/3}. \end{aligned}$$

It follows that

$$\frac{w^3}{6}\big(1-\frac{w}{2}\big) \ge \frac{2^{1/2}}{3}\Psi^{\frac{3}{2}}(w)$$

and inequality (6.2) transforms into the following one:

$$\frac{2^{1/2}}{3}\int_{x_0-\rho}^{x_0+\rho} \Psi^{\frac{3}{2}}(w(y,t))\, dy + \int_0^t \int_{x_0-\rho}^{x_0+\rho} \Psi_y^2(w)\, dy\, dt \le \int_0^t \Psi(w)\Psi_y(w)\, dt\Big|_{x_0-\rho}^{x_0+\rho}. \tag{6.3}$$

A detailed analysis of behavior of the functions satisfying inequalities of the type (6.3) is given in [1]. Let us introduce the energy functions

$$E(\rho,t) = \int_0^t \|\Psi_y(w(\cdot,t))\|^2_{L_2(B_\rho(x_0))}\, dt.$$
$$b(\rho,t) = \|\Psi(w(\cdot,t))\|^{\frac{3}{2}}_{L_{\frac{3}{2}}(B_\rho(x_0))}, \quad \overline{b}(\rho,t) = \operatorname*{ess\,sup}_{(0,t)} b(\rho,t).$$

Inequality (6.3) has the form

$$\frac{2^{1/2}}{3}\overline{b}(\rho,t) + E(\rho,t) \le I(\rho,t), \quad \text{where } I(\rho,t) := \int_0^t \Psi(w)\Psi_y(w)\, dt\Big|_{x_0-\rho}^{x_0+\rho}.$$

Using the trace-interpolation inequalities of Gagliardo-Nirenberg-Sobolev type we rewrite the last inequality in the form [1, pp.126-128]

$$E^\nu(\rho,t) \le (E+\overline{b})^\nu \le C\, t^\beta \rho^{-\alpha} E_\rho \quad \text{for } \rho(0,R), \quad E(0,t) = 0 \tag{6.4}$$

with the exponents $\nu = 6/7$, $\alpha = 4/3$, $\beta = 3/7$ and an independent of $E$ and $b$ constant $C$.

The first assertion of Theorem 2.5 follows by a straightforward integration of this inequality in the interval $(\rho, R)$, see [1, Ch.3, Proposition 1.1]. Every solution $w$ with finite energy in $\mathcal{D}$

$$\overline{b}(\rho,t)+E(\rho,t)\le \overline{b}(1/2,t^*)+E(1/2,t^*)\le 1+M$$

with the constant $M$ from estimate (4.3) possesses the following property:

$$w=0 \text{ for a.e. } |y-x_0|\le \rho(t)$$

with

$$\rho^{1+\alpha}(t)=R^{1+\alpha}-\frac{C(1+\alpha)t^{1+\beta}}{1-\nu}(1+M)^{1-\nu}.$$

The conclusion for $s(x,t)$ follows after reverting to the coordinates $(x,t)$.

The second assertion of Theorem 2.5 follows in the same way after the substitution $w\mapsto 1-w$.

## 7. Existence of self-similar solutions

**Lemma 7.1.** *The problem* (1.12)-(1.13) *has a unique solution* $\{\bar{w},\,\xi_*\}$.

*Proof.* Notice first that since the problem has the symmetry property

$$\bar{w}(\xi)+\bar{w}(-\xi)=1,$$

we may restrict the further considerations to the domain $0<\xi<\xi_*$ with the boundary conditions

$$\bar{w}(0)=\frac{1}{2},\quad \bar{w}(\xi_*)=(\bar{w})'(\xi_*)=0. \tag{7.1}$$

Let us consider the auxiliary Dirichlet problem

$$u''+\frac{\xi}{2}s'=0,\ 0<\xi<a,\quad u=\Psi(s),\quad s(0)=\frac{1}{2},\quad s(a)=0 \tag{7.2}$$

with some $a>0$. The solution of this problem depends on the parameter $a$: $s(\xi)=\Gamma(a)$. We will show that $u'(a)<0$ for all sufficiently small $a$, and the existence of a finite number $\xi_*>0$ such that $u'(a)<0$ for $a<\xi_*$ and $u'(a)\to 0$ as $a\to\xi_*$. Then $\bar{w}(x,t)=\Gamma(\xi_*)$.

It is standard to show that problem (7.2) can be solved for every $a>0$. To this end we solve first the nondegenerate problem

$$u_\varepsilon''+\frac{\xi}{2}s_\varepsilon'=0,\quad 0<\xi<a,\quad s_\varepsilon(0)=\frac{1}{2},\quad s_\varepsilon(a)=\varepsilon. \tag{7.3}$$

For every $a>0$ this problem has a unique solution $\{s_\varepsilon, u_\varepsilon=\Psi(s_\varepsilon)\}$ with the following properties:

$$\varepsilon<s_\varepsilon(\xi)<\frac{1}{2},\quad s_\varepsilon'(\xi)<0,\quad u_\varepsilon'(\xi)<0,\quad u_\varepsilon''(\xi)>0\quad \text{for } 0<\xi<a, \tag{7.4}$$

$$u_\varepsilon'(\xi_1)+\frac{\xi_1}{2}s_\varepsilon(\xi_1)-\frac{\xi_0}{2}s_\varepsilon(\xi_0)=u_\varepsilon'(\xi_0)+\frac{1}{2}\int_{\xi_0}^{\xi_1}s_\varepsilon(\xi)d\xi, \tag{7.5}$$

$$u_\varepsilon(0)-u_\varepsilon(a)+\frac{a^2}{2}s_\varepsilon(a)=-a\,u_\varepsilon'(a)+\int_0^a\xi\,s_\varepsilon(\xi)d\xi. \tag{7.6}$$

Equality (7.5) follows after integration by parts of the differential equation (7.3), (7.6) results from integration by parts of equation (7.3) multiplied by $\xi$.

Notice that equality (7.5) may be used as the definition of weak solution of problem (7.3).

Relations (7.4)–(7.6) yield boundedness of $\{u_\varepsilon\}$ in $C^1[0,a]$ and, consequently, compactness of $\{u_\varepsilon\}$ and $\{s_\varepsilon\}$ in $C[0,a]$. Relation (7.5) provides compactness of $\{u_\varepsilon\}$ in $C^1[0,a]$. Let us denote $u=\lim_{\varepsilon\to 0}u_\varepsilon$, $s=\lim_{\varepsilon\to 0}s_\varepsilon$. Passing to the limit as $\varepsilon\to 0$ in (7.4)–(7.6) we obtain

$$u=\Psi(s),\quad 0<s(\xi)<\frac{1}{2},\quad s'(\xi)<0,\quad u'(\xi)<0\quad \text{for } 0<\xi<a, \tag{7.7}$$

$$-u'(\xi_0)+\frac{\xi_1}{2}s(\xi_1)-\frac{\xi_0}{2}s(\xi_0)=-u'(\xi_1)+\frac{1}{2}\int_{\xi_0}^{\xi_1}s(\xi)d\xi, \tag{7.8}$$

$$u(0)=-a\,u'(a)+\int_0^a \xi\, s(\xi)d\xi. \tag{7.9}$$

It follows from equation (7.8) and the strong convergence of $\{s_\varepsilon\}$ that

$$u''+\frac{\xi}{2}s'=0,\ 0<\xi<a,\ s(0)=\frac{1}{2},\ s(a)=0. \tag{7.10}$$

To prove uniqueness of the solution of (7.10) we consider two different solutions $\{s_1,u_1\}$ and $\{s_2,u_2\}$. Since the differences $s=s_1-s_2$ and $u=u_1-u_2$ are continuous, there exists an interval $(\xi_0,\xi_1)$ where either $s>0$, $u>0$, or $s<0$, $u<0$, and $s(\xi_0)=s(\xi_1)=u(\xi_0)=u(\xi_1)=0$. Subtracting equations (7.8) for $\{s_1,\,u_1\}$ and $\{s_2,u_2\}$ we arrive at the equality

$$(|u_1'(\xi_0)|-|u_2'(\xi_0)|)-(|u_1'(\xi_1)|-|u_2'(\xi_1)|)=\frac{1}{2}\int_{\xi_0}^{\xi_1}\big(s_1(\xi)-s_2(\xi)\big)d\xi. \tag{7.11}$$

The simple analysis shows that the left-hand side of the last relation is non-positive, while the right-hand side is strictly positive. This contradiction proves that the solution of problem (7.10) is unique.

Equality (7.9) shows that for sufficiently small $a$ the derivative $u'(a)$ is strictly negative:

$$u'(a)=-\frac{u(0)}{a}+\frac{1}{a}\int_0^a\xi\, s(\xi)d\xi=-\frac{u(0)}{a}+\xi_*\,s(\xi_*)\leqslant -\frac{u(0)}{a}+\frac{a}{2}.$$

Here we have used the Mean Value Theorem for integrals with some $0<\xi_*\leqslant a$. If we prove that

$$\int_0^a \xi\, s(\xi)d\xi\to\infty\quad \text{as } a\to\infty, \tag{7.12}$$

then combining (7.12) with (7.9) we may find $\xi_*$ such that

$$\int_0^{\xi_*}\xi\, s(\xi)d\xi=u(0),\quad u'(\xi_*)=0. \tag{7.13}$$

This provides the existence of at least one solution of the problem (1.12)-(1.13).

Let us prove (7.12). Consider the barrier function $s_0=\frac{1}{2a}(a-\xi)$ which solves the problem

$$\big(s_0\,(1-s_0)\,s_0'\big)'+\frac{\xi}{2}s_0'=\frac{a}{4a^3}(1-a^2)\xi<0\quad \text{for } a>1,$$
$$s_0(0)=\frac{1}{2},\quad s_0(a)=0.$$

By construction $s_0(a)=0$, $s(a)=0$, and $s_0'(a)=-\frac{1}{2a}$. If $|u'(a)|>0$, then

$$|u'(a)|=\lim_{\xi\to a}s(\xi)\lim_{\xi\to a}|s'(\xi)|\quad\text{and}\quad |s'(\xi)|\to\infty\quad \text{as } \xi\to a,$$

whence $s(\xi) > s_0(\xi)$ near $\xi = a$. Let $(\xi_0, \xi_1)$ be the interval in $(0, a)$ where

$$s(\xi) < s_0(\xi), \quad \xi_0 < \xi < \xi_1, \quad s(\xi_0) = s_0(\xi_0), \quad s(\xi_1) = s_0(\xi_1).$$

Arguing as before we conclude that

$$\begin{gathered} - u_0'(\xi_0) + \frac{\xi_1}{2} s_0(\xi_1) - \frac{\xi_0}{2} s_0(\xi_0) \\ = -u_0'(\xi_1) + \frac{1}{2} \int_{\xi_0}^{\xi_1} s_0(\xi) d\xi + \frac{1}{8a^3}(1-a^2)(\xi_1^2 - \xi_0^2), \\ u_0' = \Psi(s_0). \end{gathered}$$

Proceeding as in the derivation of (7.11) and comparing $s_0$ with the solution $s$ of problem (7.2), we arrive at a contradiction. This means that

$$s(\xi) \geqslant s_0(\xi) \quad \text{for } 0 < \xi < a, \tag{7.14}$$

and (7.12) follows. In fact,

$$\int_0^a \xi\, s(\xi) d\xi \geqslant \int_0^a \xi\, s_0(\xi) d\xi = \frac{1}{2a} \int_0^a \xi\,(a-\xi) d\xi = \frac{a^2}{12} \to \infty \quad \text{as } a \to \infty.$$

To prove that the solution of problem (1.12)-(1.13) is unique we consider two possible solutions $\{\bar{w}^{(1)}, \xi_*^{(1)}\}$ and $\{\bar{w}^{(2)}, \xi_*^{(2)}\}$ on the interval $(0, \infty)$ with the boundary condition (7.1). Let us assume that $\xi_*^{(1)} < \xi_*^{(2)}$. Then the function

$$\Psi(\bar{w}) = \begin{cases} \Psi(\bar{w}^{(1)}) & \text{for } 0 < \xi < \xi_*^{(1)}, \\ 0 & \text{for } \xi_*^{(1)} < \xi < \xi_*^{(2)} \end{cases}$$

belongs to $C^1[0, \xi_*^{(2)}]$ and the pair $\{\bar{w}, \Psi(\bar{w})\}$ solves (7.2) with $a = \xi_*^{(2)}$.

By construction, the pair $\{\bar{w}^{(2)}, \Psi(\bar{w}^{(2)})\}$ is a solution of the same problem on the same interval $(0, \xi_*^{(2)})$. But the solution of this problem is unique, whence $\bar{w}^{(1)} = \bar{w}^{(2)}$ for $0 < \xi < \xi_*^{(1)}$ and $\bar{w}^{(2)} = 0$ for $\xi_*^{(1)} < \xi < \xi_*^{(2)}$. □

**Acknowledgments.** This research was supported by the Russian Science Foundation, Grant 14-17-00556.

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