On the Computation of Approximate Solutions of Fredholm Integral Equations of the First Kind with the Use of Empirical Data

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INTRODUCTION

The integral relation

$$u(y) = \int_a^b R(y,x)f(x)dx, \qquad c \le y \le d, \tag{1}$$

is often used as a mathematical model describing the response u of some device (linear system) with kernel (apparatus function) R to an input f [1, p. 225].

In practical experiments, the response at the device output is recorded at some discrete set of points in the range of the variable y, i.e., $y \in \{y_k\}$, $k = 1, \ldots, n$, $c = y_1 < y_2 < \cdots < y_n = d$, with some additive unknown error r. Thus the empirical data can be represented in the form

$$v_k = u_k + r_k, \tag{2}$$

where $v_k = v(y_k)$, $u_k = u(y_k)$, $k = 1, \ldots, n$, and the error is not related by any functional dependence to the variable y.

With regard to definition (1), the last relation can be represented in the form

$$v_k = (R_k, f) + r_k, \tag{3}$$

where the symbol (,) stands for the inner product in the Euclidean space; in this case, we have

$$(R_k, f) = \int_a^b R_k(x)f(x)dx, \tag{4}$$

$$R_k(x) = R(y_k, x), \qquad k = 1, \ldots, n. \tag{5}$$

The problem is to compute (in general, approximately) the input on the basis of such empirical data. Obviously, we deal with an approximate solution of a Fredholm integral equation of the first kind.

Note that numerous methods has been developed for the computation of approximate solutions of Fredholm integral equations of the first kind (e.g., see the bibliography in [1]). They are based on various *a priori* assumptions and principles, whose adequacy cannot be verified in most cases. The fact that the problem is ill-posed [2, p. 9] is the main reason for such a variety. The Tikhonov regularization method [2, p. 128] is most known. It is based on reducing the original equation (1) to another equation for which the computation of the approximate solution is stable under errors in the response acquisition. Omitting details, we note that the implementation of this approach results in serious difficulties in justifying the choice of the regularizing functional, including the regularization parameter [3, p. 13; 4, p. 45]. Since the input is unknown, it follows that the choice of quadrature formulas is also difficult [2, p. 152], which, obviously, has an unknown influence on

the resulting error. Since, in the case of quadrature formulas, the domain of the input is *a priori* discretized, it follows that the approximate solution is computed only at these discrete points and interpolation is required for other values of the argument. The Tikhonov regularization method is general enough to be used for the computation of stable (under errors in the original data) approximate solutions of various classes of inverse problems. Possibly, this explains the fact that, in the case of Eq. (1), it is hardly adequate in the sense of the correspondence of the computational complexity to empirical information about the input and does not admit a transparent analysis of the resulting error.

The aim of the present paper is to develop an adequate approach to the computation of approximate solutions of Eq. (1) (a basic approximate solution). This method should take account of the character of empirical information on the input and should give computational procedures that can be transparently analyzed from the viewpoint of the resulting error. Moreover, we consider the possibility of taking account of the *a priori* information on the properties of the input or *a priori* requirements on the properties of the approximate solution (an extension of the basic approximate solution).

1. REPRESENTATION OF THE BASIC APPROXIMATE SOLUTION

By physical considerations, one can assume that the variables and functions occurring in (1), including the kernel, are continuous and have a finite Euclidean norm (belong to the space L_2). The input with such properties can be represented in the form [5, p. 56 of the Russian translation]

$$f = f_1 + f_2, \tag{6}$$

where the component f_1 is represented as

$$f_1(x) = \sum_{k=1}^{n} \alpha_k R_k(x), \tag{7}$$

α_k, $k = 1, \ldots, n$, are real numbers, and the second component satisfies the orthogonality condition

$$(R_k, f_2) = 0, \qquad k = 1, \ldots, n. \tag{8}$$

If condition (8) is satisfied, then, by substituting the representation (6) into (1) and by taking account of (4), we obtain $\vec{u} = A\vec{\alpha}$, where $\vec{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)^{\mathrm{T}}$, $A = \{a_{ik}\}$, $i, k = 1, \ldots, n$, the symbol T stands for transposition, and

$$a_{ik} = (R_i, R_k)\,. \tag{9}$$

Using the notation $\vec{v} = (v_1, \ldots, v_n)^{\mathrm{T}}$ and $\vec{r} = (r_1, \ldots, r_n)^{\mathrm{T}}$, it is convenient to represent the set of relations (3) in the form $\vec{v} = A\vec{\alpha} + \vec{r}$, which implies that the empirical data on the response do not contain any information on the second component of the right-hand side of the representation (6). In other words, without additional *a priori* assumptions, the empirical data permit one to compute only approximate values of the component f_1 of the input, which is naturally referred to as the *basic approximate solution.*

It is also obvious that it is natural to use only linearly independent functions from the set $\{R_k\}$, $k = 1, \ldots, n$, in the representation (7). Thus the condition

$$f_1(x) = \sum_{i=1}^{m} \beta_i F_i(x) \tag{10}$$

should be satisfied in the general case, where $F_i = R_{k_i}$, $i = 1, \ldots, m$, m is the rank of the matrix A, the functions of the set $\{R_{k_i}\}$, $i = 1, \ldots, m$, are linearly independent, and the related indexing set is denoted by I.

Therefore, one should retain only the rows and columns with indices k_i, $i = 1, \ldots, m$, in the matrix A and only the components with these indices in the vectors $\vec{u}$ and $\vec{v}$. By the Cauchy–Schwarz inequality $u_{k_i}^2 = (f, F_i)^2 \le (f, f)\,(F_i, F_i)$, to diminish the influence of the error in the

response acquisition, it is advisable to eliminate the linearly dependent functions with minimum norm in (5).

Therefore, when eliminating rows and columns from the matrix A, one should also permute them so as to satisfy the inequalities

$$(F_i, F_i) \geq (F_{i+1}, F_{i+1}), \qquad i = 1, \ldots, m-1; \tag{11}$$

i.e., the diagonal entries of the resulting matrix should be arranged in decreasing order.

Obviously, the elimination of some empirical data is, in a sense, filtering; moreover, the resulting vector of response values admits the representation

$$\vec{v}_1 = A_1\vec{\beta} + \vec{r}_1, \tag{12}$$

where the unit index implies that the matrix and vectors, including the vector of error in the response acquisition, are transformed in accordance with the above-described procedure.

Obviously, since the matrix on the right-hand side in (12) is nondegenerate, it follows that empirical data determine a unique [similar to (10)] representation of the basic approximate solution

$$\hat{f}_1(x) = \sum_{k=1}^{m} b_k F_k(x), \tag{13}$$

where

$$A_1\vec{b} = \vec{v}_1. \tag{14}$$

If the acquisition errors are zero, then the vector $\vec{b} = (b_1, \ldots, b_n)^{\mathrm{T}}$ of parameters coincides with the parameter vector of the representation (10) (neglecting computation errors).

2. ON THE COMPUTATION OF THE BASIC APPROXIMATE SOLUTION

One can readily see that the main problem is to find the numerical values of the parameters of the representation (13) on the basis of equality (14). Therefore, one should find the rank of the matrix A with entries (9) and the set of indices of linearly independent functions from the set (5), which satisfy condition (11); this would allow one to form the matrix and vector occurring on the right-hand side in equality (14).

Let C stand for the matrix obtained from A by permuting rows and columns in decreasing order of the diagonal entries, and let $\vec{q}$ be the vector with the corresponding rearrangement of components in the vector $\vec{v}$.

It readily follows from definition (9) that the matrix C is symmetric and negative definite, and its determinant is the Gram determinant [5, p. 68 of the Russian translation]. The upper corner submatrices [6, p. 7] have the same properties. Therefore, by successively computing the upper corner minors, one can find the zero minor of the upper submatrix of the minimum dimension n_1. On eliminating the row and column with the corresponding index from C, one can proceed to the computation of the resulting upper minors until the next of them vanishes. This yields the index n_2 of the following eliminated row and column of the original matrix, and so on. One should simultaneously eliminate the corresponding entries in the vector $\vec{q}$.

For the successive computation of minors, it is advisable to use the method of reducing a matrix to the Gauss upper triangular form [7, p. 96], which, together with the Gaussian inverse substitution and with appropriate transformations of the vector of the right-hand side in Eq. (14) permits one to find the solution of this equation.

By (9), the computation of the entries of the original matrix is reduced to the numerical integration, which can be performed with the use of quadrature formulas [4, p. 223] most precise for each pair of functions in the set (5).

3. EXTENSIONS OF THE BASIC APPROXIMATE SOLUTION

The aim of the extension of the basic approximate solution is to define the component f_2 occurring in the representation (6) in the form of some estimate $\hat{f}_2$. As was mentioned above, this is

possible only with the use of some *a priori* assumptions on properties of the input. In this case, it is natural to require that this component of the extended approximate solution $\hat{f} = \hat{f}_1 + \hat{f}_2$ of Eq. (1) be orthogonal to the basic approximate solution of the form (13), whose parameters satisfy Eq. (14).

We set

$$\hat{f}(x) = \sum_{k=1}^{m} z_k G_k(x), \tag{15}$$

where the z_k are real parameters and G_k is a set of real-valued linearly independent functions.

Let us derive conditions ensuring that the projection of (15) on the lineal [5, p. 15 of the Russian translation] of the form (10) is given by relations (13) and (14).

Obviously, in this case, we have the relations

$$\left(F_k,\ \sum_{i=1}^{m} [z_i G_i - b_i F_i]\right) = 0, \qquad k = 1, \ldots, m,$$

which, together with (14), readily give an equation for the parameters of the representation (15):

$$P\vec{z} = \vec{v}_1, \tag{16}$$

where $\vec{z} = (z_1, \ldots, z_n)^{\mathrm{T}}$ and $P = \{p_{ik}\} = (F_i, G_k)$, $i, k = 1, \ldots, m$.

Under the above-mentioned conditions, the matrix occurring in (16) is nonsingular. Therefore, the vector of parameters of the sum (15) satisfies the relation $\vec{z} = P^{-1}\vec{v}_1$.

Then, by taking account of relations (2) and (3), one can reduce the representation (15) to the form

$$\hat{f} = \hat{f}_r + \hat{f}_f, \tag{17}$$

where the first term is determined by the error in the response acquisition, and the second term (the regular part) is determined by the input:

$$\hat{f}_f(x) = \int_a^b B(x, s) f(s) ds, \qquad B(x, s) = \sum_{i,k=1}^{m} p_{ik}^{-1} G_k(x) R_i(s), \tag{18}$$

and the p_{ik}^{-1} are the entries of the matrix P^{-1}.

Obviously, when choosing the function set in the representation (15), one should try to make the regular part of the approximate solution as close as possible to the unknown input; i.e., the relation $\hat{f}_f(x) = f(x)$, $a \le x \le b$, must be valid in the ideal case.

Obviously, this identity is valid if and only if the kernel in the first relation (18) has the unit eigenvalue and the corresponding eigenfunction coincides with the input.

By using the definition of the entries of the matrix in relation (16) and the representation of the kernel $B(x, s)$ in (18), one can readily prove the identities

$$G_k(x) = \int_a^b B(x, s) G_k(s) ds, \qquad k = 1, \ldots, m. \tag{19}$$

In other words, each function occurring in the representation (15) and hence the representation in the whole are eigenfunctions of the kernel $B(x, s)$ of the form (18) corresponding to the unit eigenvalue.

This fact allows one to use the *a priori* information on properties of the input, which is often provided in applied investigations. Therefore, as basis functions of the lineal of the form (15), one should use functions such that the norm of components of the inputs orthogonal to it forms a small part of the total norm.

Note also that the computation of the entries of the matrix in Eq. (16) is related to the choice of appropriate (for given function pairs) quadrature formulas, and the solution of this equation does not go beyond the well-developed methods of linear algebra.

We should note one more quite general method for the extension of the basic approximate solution. It is based on the use of *a priori* conditions in the form of variational principles

$$W(f) = \text{extr}, \qquad (f, R_{k_i}) = v_{k_i}, \qquad i = 1, \ldots, m, \tag{20}$$

where W is some nonnegative functional whose extremal is determined under given constraints (the isoperimetric problem [8, p. 221]).

Note that if the squared Euclidean norm of the approximation computed for the input in (1) on the basis of empirical data is to be minimized, then the solution of the variational problem (20) gives the representation (13) and Eq. (14).

By way of example, we present the functional in the form of the squared norm of the first-order derivative

$$W(f) = \int_a^b \left(f^{(1)}(x)\right)^2 dx. \tag{21}$$

The use of the well-known approaches [8, p. 221] to isoperimetric variational problems allows one to obtain a function providing the minimum of the functional (21) under constraints of the form (15) occurring in (20), where the basic functions $G_k(x)$ satisfy the differential equations

$$G_k^{(2)}(x) = R_{i_k}(x), \qquad k = 1, \ldots, m. \tag{22}$$

In the computation of solutions of equations of the form (22), it is natural to use the boundary conditions $G_k(a) = R_{i_k}(a)$ and $G_k(b) = R_{i_k}(b)$, $k = 1, \ldots, m$. The following step is to solve a system of equations of the form (16); for this purpose, one should first compute the corresponding entries of the matrix.

Obviously, the implementation of all computations, including quadratures of integrals, does not face any essential difficulties.

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