Goldbach's Ternary Problem Involving Prime Numbers Expressible by Given Quadratic Forms

S. A. Gritsenko*

Belgorod State University

Abstract—In this paper, we solve Goldbach's ternary problem involving primes expressible by given primitive positive definite binary quadratic forms whose discriminants coincide with the discriminants of imaginary quadratic fields in which quadratic forms split into linear multipliers.

Key words: *Goldbach's ternary problem, binary quadratic form, imaginary quadratic field, arithmetic progression, simple ideal, Dirichlet series.*

INTRODUCTION

In 1937, Vinogradov solved Goldbach's ternary problem by obtaining an asymptotic formula for the number of solutions of the equation

$$p_1 + p_2 + p_3 = N \qquad (1)$$

in primes p_1, p_2, p_3 with N odd.

In the present paper, we obtain an asymptotic formula for the number of solutions of Eq. (1) in primes p_1, p_2, p_3 such that each prime p_i, $i = 1, 2, 3$ can be expressed by a binary quadratic form.

Note that if these quadratic forms are one-class, then the problem can be reduced, essentially, to the solution of Goldbach's ternary problem involving primes lying in certain arithmetic progressions. For the case in which the differences of these arithmetic progressions are constants or increase with N not too fast, the solution of such a problem does not differ significantly from that of the classical Goldbach ternary problem. The same applies to the problem of obtaining an asymptotic formula for the number of solutions of (1) in primes expressible by all quadratic forms of given discriminants.

If quadratic forms representing primes, are multiclass, the question of expressing positive integers by such forms cannot be reduced to the question of whether these numbers belong to some classes of residues. (in this connection, see [1]).

Therefore, to solve the problem under consideration, one must have some information about the distribution of primes expressible by quadratic forms in arithmetic progressions as well as estimates of some special trigonometric sums.

In what follows, by quadratic forms we shall mean primitive positive definite binary quadratic forms whose discriminants coincide with the discriminants of imaginary quadratic fields in which these forms split into linear multipliers.

The discriminants of quadratic forms are assumed to be constant everywhere, although it can be seen from the proofs of the main theorem and the lemmas that they may increase as the parameter N increases.

In what follows, the following notation will be used:

- p, p_1, p_2 are primes;
- P are simple ideals of an imaginary quadratic field;

*E-mail: gritsenko@bsu.edu.ru

- $N(A)$ is the norm of an integer ideal A of an imaginary quadratic field;

- $\chi(q;k,l) = \begin{cases} 1 & \text{if } q \equiv l \pmod{k}, \\ 0 & \text{otherwise;} \end{cases}$

- $\chi_i(l)$ is a character of a quadratic field in which the quadratic form Q_i splits into linear multipliers, $i = 1, 2, 3$;

- $S_Q(\alpha) = \sum' e^{2\pi i \alpha p}$, where the prime stands for summation over primes expressible by a quadratic form Q;

- $\sum\limits_{a=0}^{b-1}{}^{*}$ denotes summation over a's coprime to b in the specified limits.

The main results of the paper are the following theorems.

Theorem 1. *Suppose that $J(N, Q_1, Q_2, Q_3)$ is the number of solutions of Eq. (1) in primes p_1, p_2, p_3 expressible by quadratic forms Q_1, Q_2, Q_3 with discriminants $-D_1$, $-D_2$, $-D_3$, respectively, h_1, h_2, h_3 are the numbers of classes of equivalent forms with discriminants $-D_1$, $-D_2$, $-D_3$, respectively. Then, for any constant $c > 0$, the following formula holds:*

$$J(N, Q_1, Q_2, Q_3) = \sigma(N, D_1, D_2, D_3) I(N) + O(N^2 \log^{-c} N),$$

where

$$I(N) = \sum_{\substack{n_1+n_2+n_3=N \\ 3 \le n_1, n_2, n_3 \le N}} \frac{1}{\log n_1 \, \log n_2 \, \log n_3} \sim \frac{N^2}{2 \log^3 N},$$

$$\begin{aligned} \sigma(N, D_1, D_2, D_3) = {} & \frac{1}{h_1 h_2 h_3} \sum_{q=1}^{\infty} \frac{1}{\varphi^3(q)} \sum_{a=0}^{q-1}{}^{*} e^{-2\pi i a N/q} \\ & \times \left(\mu(q) + \chi(q; D_1, 0) \chi_1(a) \mu\left(\frac{q}{D_1}\right) \chi_1\left(\frac{q}{D_1}\right) \tau_{\chi_1} \right) \\ & \times \left(\mu(q) + \chi(q; D_2, 0) \chi_2(a) \mu\left(\frac{q}{D_2}\right) \chi_2\left(\frac{q}{D_2}\right) \tau_{\chi_2} \right) \\ & \times \left(\mu(q) + \chi(q; D_3, 0) \chi_3(a) \mu\left(\frac{q}{D_3}\right) \chi_3\left(\frac{q}{D_3}\right) \tau_{\chi_3} \right) \end{aligned}$$

and χ_1, χ_2, χ_3 are the characters of the imaginary quadratic fields with discriminants $-D_1$, $-D_2$, $-D_3$, respectively.

Theorem 2. *Suppose that $J(N, Q_1, Q_2, Q_3)$ is the number of solutions of Eq. (1) in primes p_1, p_2, p_3 expressible by quadratic forms Q_1, Q_2, Q_3 with discriminant $-D$ and h is the number of classes of equivalent forms with discriminant $-D$. Then, for any constant $c > 0$, the following formula holds:*

$$J(N, Q_1, Q_2, Q_3) = \sigma(N, D) I(N) + O(N^2 \log^{-c} N),$$

where

$$I(N) = \sum_{\substack{n_1+n_2+n_3=N \\ 3 \le n_1, n_2, n_3 \le N}} \frac{1}{\log n_1 \, \log n_2 \, \log n_3} \sim \frac{N^2}{2 \log^3 N},$$

$$\sigma(N, D) = \sigma_0(N) + \sigma_1(N, D) + \sigma_2(N, D) + \sigma_3(N, D),$$

$$\sigma_0(N) = \frac{1}{h^3}\sum_{q=1}^{\infty}\frac{\mu(q)\gamma_0(N,q)}{\varphi^3(q)}, \qquad \sigma_1(N,D) = \frac{3}{h^3}\sum_{\substack{q_2=1\\(q_2,D)=1}}^{\infty}\frac{\mu^2(Dq_2)\gamma_1(N,Dq_2)}{\varphi^3(q_2)},$$

$$\sigma_2(N,D) = \frac{3}{h^3}\sum_{\substack{q_2=1\\(q_2,D)=1}}^{\infty}\frac{\mu(Dq_2)\gamma_2(N,Dq_2)}{\varphi^3(q_2)}, \qquad \sigma_3(N,D) = \frac{1}{h^3}\sum_{\substack{q_2=1\\(q_2,D)=1}}^{\infty}\frac{\gamma_3(N,Dq_2)}{\varphi^3(q_2)},$$

$$\gamma_0(N,q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e^{-2\pi iaN/q}, \qquad \gamma_1(N,q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} S(a,q)e^{-2\pi iaN/q},$$

$$\gamma_2(N,q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} S^2(a,q)e^{-2\pi iaN/q}, \qquad \gamma_3(N,q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} S^3(a,q)e^{-2\pi iaN/q};$$

here

$$S(a,q) = \sum_{\substack{l=1\\(l,q)=1}}^{q}\chi_1(l)e^{2\pi ial/q}$$

and χ_1 is the character of the imaginary quadratic field with discriminant $-D$.

The proofs are carried out by the circular method. Besides, an essential role is played by the functional equation of the Dirichlet series of special form from [2] on which the asymptotic formula from [3] and the estimate from [4] are based.

To prove Theorems 1 and 2, we need the following lemmas.

1. LEMMAS

Lemma 1. *Suppose that $(a, D_1q_2) = 1$, $(D, q_2) = 1$, $D \mid D_1$; moreover, each prime divisor D_1 divides D. Then the following identity holds:*

$$S(a, D_1q_2) = \begin{cases}\chi_1(a)\chi_1(q_2)\mu(q_2)\tau_{\chi_1} & \text{if } D_1 = D,\\ 0 & \text{if } D_1 \neq D,\end{cases}$$

where $\tau_{\chi_1} = \sum_{l=1}^{D}\chi_1(l)e^{2\pi il/D}$ is the Gauss sum.

Proof. Since the character χ_1 is real, we have

$$S(a, D_1q_2) = \chi_1(a)S(1, D_1q_2).$$

Further,

$$S(1, D_1q_2) = \sum_{\substack{l_1=1\\(l_1,D_1)=1}}^{D_1}\sum_{\substack{l_2=1\\(l_2,q_2)=1}}^{q_2}\chi_1(l_1q_2 + l_2D_1)e^{2\pi il_1/D_1}e^{2\pi il_2/q_2}$$

$$= \chi_1(q_2)\mu(q_2)\sum_{l_1=1}^{D}\sum_{l_2=0}^{D_1/D-1}\chi_1(l + Dl_2)e^{2\pi il_1/D_1}e^{2\pi il_2/(D_1/D)}.$$

Since the sum over l_2 is equal to 1 if $D_1 = D$ and zero otherwise, we obtain the proof of Lemma 1. □

Lemma 2. *Suppose that* $(D, q_2) = 1$. *Then the following identities hold:*

$$\gamma_1(N, Dq_2) = D\mu(q_2)\chi_1(N)\gamma_0(N, q_2),$$
$$\gamma_2(N, Dq_2) = \chi_1(-1)D\mu^2(q_2)\gamma_0(N, D)\gamma_0(N, q_2),$$
$$\gamma_3(N, Dq_2) = \chi_1(-1)D^2\mu(q_2)\chi_1(N)\gamma_0(N, q_2).$$

Proof. Let us use Lemma 1. We have

$$\gamma_1(N, Dq_2) = \chi_1(q_2)\mu(q_2)\tau(\chi_1) \sum_{\substack{a=1\\ a,Dq_2)=1}}^{Dq_2} \chi_1(a)e^{-2\pi iaN/(Dq_2)}$$
$$= \chi_1(q_2)\mu(q_2)\tau(\chi_1) \sum_{\substack{a_1=1\\ a,D)=1}}^{D} \sum_{\substack{a_2=1\\ a_2,q_2)=1}}^{q_2} \chi_1(a_1q_2 + a_2D)e^{-2\pi ia_1N/D}e^{-2\pi ia_2N/q_2}$$
$$= \chi_1^2(q_2)\mu(q_2)\tau(\chi_1)\gamma_0(N, q_2)\sum_{a=1}^{D}\chi_1(a)e^{-2\pi iaN/D}.$$

Let us apply the well-known identity

$$\sum_{a=1}^{D}\chi_1(a)e^{-2\pi iaN/D} = \chi_1(-1)\chi_1(N)\tau_{\chi_1};$$

then we obtain

$$\gamma_1(N, Dq_2) = \chi_1^2(q_2)\mu(q_2)\chi_1(-1)\tau_{\chi_1}^2\chi_1(N)\gamma_0(N, q_2).$$

Further, $\chi_1^2(q_2) = 1$ and $\chi_1(-1)\tau_{\chi_1}^2 = D$, because χ_1 is a real character and $(q_2, D) = 1$. Thus, we have obtained an identity for $\gamma_1(N, Dq_2)$. Let us also derive an identity for $\gamma_2(N, Dq_2)$.

By Lemma 1, we have

$$\gamma_2(N, Dq_2) = \mu^2(q_2)\tau_{\chi_1}^2\gamma_0(N, Dq_2).$$

Now, the required identity follows from the fact that the function $\gamma_0(N, q)$ is multiplicative with respect to q and from equality $\tau_{\chi_1}^2 = \chi_1(-1)D$.

Let us derive an identity for $\gamma_3(N, Dq_2)$. We again apply Lemma 1, obtaining

$$\gamma_3(N, Dq_2) = \tau_{\chi_1}^2\gamma_1(N, Dq_2) = \chi_1(-1)D^2\mu(q_2)\chi_1(N)\gamma_0(N, q_2).$$

Lemma 2 is proved. □

Lemma 3. *The following identities hold:*

$$\sigma_0(N) = \frac{1}{h^3}\prod_p\left(1 - \frac{\gamma_0(N, p)}{\varphi^3(p)}\right), \qquad \sigma_1(N, D) = \frac{3D\mu^2(D)\chi_1(N)}{h^3\varphi^3(D)}\prod_{p\nmid D}\left(1 - \frac{\gamma_0(N, p)}{\varphi^3(p)}\right),$$
$$\sigma_2(N, D) = \frac{3D\mu(D)\chi_1(-1)\gamma_0(N, D)}{h^3\varphi^3(D)}\prod_{p\nmid D}\left(1 - \frac{\gamma_0(N, p)}{\varphi^3(p)}\right),$$
$$\sigma_3(N, D) = \frac{D^2\chi_1(-1)\chi_1(N)}{h^3\varphi^3(D)}\prod_{p\nmid D}\left(1 - \frac{\gamma_0(N, p)}{\varphi^3(p)}\right).$$

Proof. Lemma 3 immediately follows from Lemma 2 and the fact that the function $\gamma_0(N, q)$ is multiplicative with respect to q. □

Lemma 4. *Suppose that* $(l,q)=1$, $1\le q\le \log^{c_1} x$, *and* $\mathscr{C}$ *is the class of ideals. Then the following asymptotic formula holds*:

$$\pi_1(x;q,l,\mathscr{C})=\sum_{l=1}^{q}{}^{*}\,\frac{1+\chi(q;D,0)\chi_1(l)}{h\varphi(q)}\operatorname{Li}x+O(Ne^{-c_0(\log N)^{\gamma}/2}),$$

where $c_0=c_0(c_1)>0$, $\gamma=1/(20c_1)$.

Proof. The proof is given in [3]. □

Lemma 5. *Suppose that* $(a,q)=1$, $\alpha=a/q+z$, $q\le\log^{c_1}N$, $|z|\le 1/(q\tau)$, *and* $\tau=N\log^{-c_2}N$. *Then, for* $S_Q(\alpha)$, *the following formula holds*:

$$S_Q(\alpha)=\sum_{l=1}^{q}{}^{*}\,\frac{1+\chi(q;D,0)\chi_1(l)}{h\varphi(q)}e^{2\pi i a l/q}M(z)+O(Ne^{-c_0(\log N)^{\gamma}/2}),$$

where c_0 *and* γ *are the constants from the lemma and*

$$M(z)=\sum_{n=3}^{N}\frac{e^{2\pi i z n}}{\log n}\,.$$

Proof. By assumption, the form Q can be expressed as

$$Q(x,y)=\frac{N(x\omega_1+y\omega_2)}{N(A)}\,,$$

where ω_1,ω_2 is the basis of an integer ideal A.

Suppose that A lies in the class of ideals $\mathscr{A}$ and B is an arbitrary ideal from the class $\mathscr{A}^{-1}$. Then $AB=(\xi_B)$ is the principal ideal, $\xi_B\in A$.

The mapping $B\to\xi_B$ induces a bijection between the proper ideals from the class $\mathscr{A}^{-1}$ and the classes of equivalent numbers from the ideal A (two numbers *are equivalent* if their ratio is ±1). Since

$$\frac{N(\xi_B)}{N(A)}=N(B),$$

it follows that only those primes p that are the norms of simple ideals from the class $\mathscr{A}^{-1}$ can be expressed by the form Q.

Hence we have the equality

$$S_Q(\alpha)=\sum_{\substack{P\in\mathscr{A}^{-1}\\ N(P)\le N}}e^{2\pi i\alpha N(P)}+O\big(\sqrt{N}\,\big)=S_{1Q}(\alpha)+O\big(\sqrt{N}\,\big).$$

The remainder $O(\sqrt{N}\,)$ is due to the fact that there exist $O(\sqrt{N}\,)$ simple ideals whose norms are equal to $p^2\le N$.

Now we consider the sum $S_{1Q}(\alpha)$. We have

$$S_{1Q}(\alpha)=S_{1Q}\Big(\frac{a}{q}+z\Big)=\sum_{l=1}^{q}{}^{*}\,e^{2\pi i a l/q}T_l(z),$$

where

$$T_l(z)=\sum_{\substack{P\in\mathscr{A}^{-1}\\ N(P)\le N\\ N(P)\equiv l\ (\mathrm{mod}\, q)}}e^{2\pi i z N(P)}.$$

Applying to $T_l(z)$ the partial summation formula, we obtain

$$T_l(z) = -\int_{\sqrt{N}}^{N} \big(\pi_1(u; q, l, \mathscr{A}^{-1}) - \pi_1(\sqrt{N}; q, l, \mathscr{A}^{-1})\big)\, de^{2\pi i z u} + \pi_1(N; q, l, \mathscr{A}^{-1}) e^{2\pi i z N} + O\big(\sqrt{N}\,\big),$$

where

$$\pi_1(u; q, l, \mathscr{A}^{-1}) = \sum_{\substack{P \in \mathscr{A}^{-1} \\ N(P) \le u \\ N(P) \equiv l \ (\mathrm{mod}\, q)}} 1.$$

Integrating by parts and using Lemma 4, we find

$$T_l(z) = \frac{1 + \chi(q; D, 0)\chi_1(l)}{h\varphi(q)} \int_3^N \frac{e^{2\pi i z u}}{\log u}\, du + O(N e^{-c_0 (\log N)^{\gamma}/2}).$$

The assertion of the lemma now follows from the well-known equality

$$M(z) = \int_3^N \frac{e^{2\pi i z u}}{\log u}\, du + O(1). \quad \square$$

Lemma 6. *Suppose that*

$$a(n) = \sum_{\substack{A \\ N(A) = n}} X(A),$$

where X is an arbitrary nonprincipal character of the group of classes of ideals. Suppose that $c > 0$ is an arbitrary constant. Then there exist positive constants c_1 and c_2 such that, for $\alpha = a/q + z$, $|z| \le 1/(q\tau)$, $(a, q) = 1$, $\tau = N(\log N)^{-c_2}$, $(\log N)^{c_1} < q \le \tau$, the following estimate holds:

$$\sum_{n=1}^{N} \Lambda(n) a(n) e^{2\pi i \alpha n} = O\big(N(\log N)^{-c}\big).$$

Proof. The proof is given in [4]. $\square$

Corollary 1. *Suppose that $c > 0$ is an arbitrary constant. Then there exist positive constants c_1 and c_2 such that, for $\alpha = a/q + z$, $|z| \le 1/(q\tau)$, $(a, q) = 1$, $\tau = N(\log N)^{-c_2}$, $(\log N)^{c_1} < q \le \tau$, the following estimate holds:*

$$S_Q(\alpha) = O\big(N(\log N)^{-c}\big).$$

Proof. The following relations are valid:

$$S_Q(\alpha) = \sum_{\substack{P \in \mathscr{A}^{-1} \\ N(P) \le N}} e^{2\pi i \alpha N(P)} + O\big(\sqrt{N}\,\big) = \frac{1}{h} \sum_{X} \overline{X}(\mathscr{A}) \sum_{N(P) \le N} X(P) e^{2\pi i \alpha N(P)} + O\big(\sqrt{N}\,\big),$$

where X ranges over all characters of the group of classes of ideals and h is the order of the group of classes of ideals.

Further,

$$\sum_{X} \overline{X}(\mathscr{A}) \sum_{N(P) \le N} X(P) e^{2\pi i \alpha N(P)} = \sum_{p \le N} e^{2\pi i \alpha p} + \sum_{p \le N} \chi_1(p) e^{2\pi i \alpha p} + \sum_{X \ne X_0} \overline{X}(\mathscr{A}) \sum_{N(P) \le N} X(P) e^{2\pi i \alpha N(P)},$$

where χ_1 is the character of the quadratic field and X_0 is the principal character of the group of classes of ideals.

The estimate

$$\sum_{X\neq X_0}\overline{X}(\mathscr{A})\sum_{N(P)\leq N}X(P)e^{2\pi i\alpha N(P)}=O\big(N(\log N)^{-c}\big)$$

follows from Lemma 6.

The estimate

$$\sum_{p\leq N}e^{2\pi i\alpha p}=O\big(N(\log N)^{-c}\big)$$

is well known (see, for example, [5, Chap. 10]). The estimate

$$\sum_{p\leq N}\chi_1(p)e^{2\pi i\alpha p}=O\big(N(\log N)^{-c}\big)$$

also holds and can be obtained, essentially, in the same way as the estimate of the sum $\sum_{p\leq N}e^{2\pi i\alpha p}$, because the modulus of the Dirichlet character χ_1 is a constant (nonincreasing as N increases). □

2. PROOF OF THE MAIN THEOREM

Proof of Theorem 1. Suppose that $\tau=N\log^{-c_2}N$ and $c_2>1$. We write $J(N,Q_1,Q_2,Q_3)$ in integral form:

$$J(N,Q_1,Q_2,Q_3)=\int_{-1/\tau}^{1-1/\tau}S_{Q_1}(\alpha)S_{Q_2}(\alpha)S_{Q_3}(\alpha)e^{-2\pi i\alpha N}\,d\alpha.$$

Let us define the sets E_1 and E_2:

$$E_1=\left\{\alpha\in\left[-\frac{1}{\tau},1-\frac{1}{\tau}\right):\left|\alpha-\frac{a}{q}\right|\leq\frac{1}{q\tau},\ (a,q)=1,\ q\leq\log^{c_1}N\right\},$$

$$E_2=\left[-\frac{1}{\tau},1-\frac{1}{\tau}\right)\setminus E_1.$$

Choose the parameters c_1 and c_2 so that $c+1<c_1,\ 2c_1<c_2$; moreover, c_1 and c_2 satisfy the assumptions of Lemma 6. Then if N is a sufficiently large number, then the sets E_1 and E_2 consist of nonintersecting intervals and

$$J(N,Q_1,Q_2,Q_3)=J_1(N,Q_1,Q_2,Q_3)+J_2(N,Q_1,Q_2,Q_3),$$

where

$$J_i(N,Q_1,Q_2,Q_3)=\int_{E_i}S_{Q_1}(\alpha)S_{Q_2}(\alpha)S_{Q_3}(\alpha)e^{-2\pi i\alpha N}\,d\alpha,\qquad i=1,2.$$

By Corollary 1, we have

$$J_2(N,Q_1,Q_2,Q_3)=O\big(\pi(N)\max_{\alpha\in E_2}|S_{Q_1}(\alpha)|\big)=O(N^2\log^{-c}N).$$

Suppose that $\alpha\in E_1$. Applying Lemmas 1 and 5, we obtain

$$\begin{aligned}J_1(N,Q_1,Q_2,Q_3)=\frac{1}{h_1h_2h_3}\sum_{q\leq\log^{c_1}N}\frac{1}{\varphi^3(q)}&\sideset{}{^*}\sum_{a=0}^{q-1}e^{-2\pi iaN/q}\\&\times\left(\mu(q)+\chi(q;D_1,0)\chi_1(a)\mu\left(\frac{q}{D_1}\right)\chi_1\left(\frac{q}{D_1}\right)\tau_{\chi_1}\right)\\&\times\left(\mu(q)+\chi(q;D_2,0)\chi_2(a)\mu\left(\frac{q}{D_2}\right)\chi_2\left(\frac{q}{D_2}\right)\tau_{\chi_2}\right)\end{aligned}$$

$$\times \left(\mu(q) + \chi(q; D_3, 0)\chi_3(a)\mu\left(\frac{q}{D_3}\right)\chi_3\left(\frac{q}{D_3}\right)\tau_{\chi_3}\right)$$
$$\times \int_{-1/(q\tau)}^{1/(q\tau)} M^3(z)e^{-2\pi izN}\,dz.$$

Following arguments from [5, Chap. 10], we can obtain the equality

$$\int_{-1/(q\tau)}^{1/(q\tau)} M^3(z)e^{-2\pi izN}\,dz = \int_{-1/2}^{1/2} M^3(z)e^{-2\pi izN}\,dz + O\big(N^2(\log N)^{-2c_2+2c_1}\big)$$
$$= I(N) + O\big(N^2(\log N)^{-2c_2+2c_1}\big).$$

Finally, trivially estimating the sum over $q > \log^{c_1} N$, we obtain the assertion of Theorem 1. □

Proof of Theorem 2. Theorem 2 is proved in the same way, the only exception being that the sum over a is calculated with the help of Lemmas 2 and 3. □

3. LOWER BOUNDS FOR THE SUMS OF SINGULAR SERIES

With regard to the sum of the singular series from Theorem 1, we restrict ourselves to the following assertion.

Proposition 1. *Suppose that the numbers D_1, D_2, and D_3 are sufficiently large and N is an odd number. Then the following inequality holds:*

$$\sigma(N, D_1, D_2, D_3) > \frac{1}{2h_1h_2h_3}\,.$$

Proof. Let us remove the brackets in the product

$$\left(\mu(q) + \chi(q; D_1, 0)\chi_1(a)\mu\left(\frac{q}{D_1}\right)\chi_1\left(\frac{q}{D_1}\right)\tau_{\chi_1}\right)$$
$$\times \left(\mu(q) + \chi(q; D_2, 0)\chi_2(a)\mu\left(\frac{q}{D_2}\right)\chi_2\left(\frac{q}{D_2}\right)\tau_{\chi_2}\right)$$
$$\times \left(\mu(q) + \chi(q; D_3, 0)\chi_3(a)\mu\left(\frac{q}{D_3}\right)\chi_3\left(\frac{q}{D_3}\right)\tau_{\chi_3}\right).$$

We obtain eight summands, which correspond to eight singular series. The first summand is equal to $\mu(q)$; it corresponds to the series

$$\sigma_1(N) = \frac{1}{h_1h_2h_3}\sum_{q=1}^{\infty} \frac{\mu(q)\gamma_0(q, N)}{\varphi^3(q)}\,.$$

It is well known that $\sigma_1(N) > 1/(h_1h_2h_3)$ (see, for example, [5, Chap. 10]).

Let us find an upper bound for the sums of the other series. Since all the estimates are obtained identically, we restrict ourselves to the series

$$\sigma_8(N, D_1, D_2, D_3) = \frac{1}{h_1h_2h_3}\sum_{q=1}^{\infty}\sideset{}{^*}\sum_{a=0}^{q-1} e^{-2\pi iaN/q}\,\frac{\chi(q; D_1, 0)\chi(q; D_2, 0)\chi(q; D_3, 0)}{\varphi^3(q)}$$
$$\times \chi_1(a)\chi_2(a)\chi_3(a)\mu\left(\frac{q}{D_1}\right)\mu\left(\frac{q}{D_2}\right)\mu\left(\frac{q}{D_3}\right)$$
$$\times \chi_1\left(\frac{q}{D_1}\right)\chi_2\left(\frac{q}{D_2}\right)\chi_3\left(\frac{q}{D_3}\right)\tau_{\chi_1}\tau_{\chi_2}\tau_{\chi_3}.$$

Using the root estimates for the Gauss sums, we find that there exists an absolute constant $c_3 > 0$ such that

$$|\sigma_8(N, D_1, D_2, D_3)| < \frac{c_3\sqrt{D_1 D_2 D_3}}{h_1 h_2 h_3 [D_1, D_2, D_3]^2},$$

where $[D_1, D_2, D_3]$ is the least common multiple of the numbers D_1, D_2, D_3.

If $D_1 \le D_2 \le D_3$, then $\sqrt{D_1 D_2 D_3} \le D_3^{3/2}$, $[D_1, D_2, D_3]^2 \ge D_3^2$; therefore, the following inequality holds:

$$|\sigma_8(N, D_1, D_2, D_3)| < \frac{c_3}{h_1 h_2 h_3 \sqrt{\max(D_1, D_2, D_3)}} \le \frac{1}{14 h_1 h_2 h_3}$$

if

$$\min(D_1, D_2, D_3) \ge 196 c_3^2.$$

Using similar procedures with regard to the six remaining singular series, we obtain our assertion. □

Now consider the singular series from Theorem 2.

Proposition 2. *If N is an even number, then*

$$\sigma_0(N) = \sigma_1(N, D) = \sigma_2(N, D) = \sigma_3(N, D) = 0.$$

Proof. If D is an even number, then, as is known, D is divisible by 4; hence $\sigma_1(N) = \sigma_2(N) = 0$. Also, $\chi_1(N) = 0$, i.e., $\sigma_3(N) = 0$. Finally, it is also well known that $\sigma_0(N) = 0$ for an even N.

But if D is an odd number, then, for an even N, we have

$$\prod_{p\nmid D}\left(1 - \frac{\gamma_0(N,p)}{\varphi^3(p)}\right) = 0, \qquad \sigma_0(N) = \sigma_1(N, D) = \sigma_2(N, D) = \sigma_3(N, D) = 0. \quad \square$$

Proposition 3. *Suppose that N is an odd number and $-D = \delta_F$ is the discriminant of a quadratic form which is not one-class and expands into linear multipliers in the imaginary quadratic field with discriminant δ_F. Suppose that $D > 15$. Then the inequality $\sigma(N, D) > 2/(5h^3)$ holds.*

Proof. Since $-D$ is the discriminant of a quadratic field, it suffices to consider the following cases.

Case 1. $D = 8d$, where d is an odd square-free number.

Then $\mu(D) = 0$, $\sigma_1(N, D) = \sigma_2(N, D) = 0$, and hence $\sigma_0(N, D) = \sigma_0(N) + \sigma_3(N, D)$.

If $(N, D) > 1$, then $\chi_1(N) = 0$, $\sigma_3(N, D) = 0$; therefore, in this case, $\sigma(N, D) = \sigma_0(N)$.

Suppose that $(N, D) = 1$; then

$$\sigma(N, D) \ge \frac{1}{h^3}\left(2\prod_{p\mid d}\left(1 + \frac{1}{(p-1)^3}\right) - \prod_{p\mid d}\frac{p^2}{(p-1)^3}\right)\prod_{p\nmid D}\left(1 - \frac{\gamma_0(N,p)}{\varphi^3(p)}\right) \ge \frac{1}{2}\sigma_0(N),$$

because, for any $p \ge 3$, the following inequality holds:

$$1 + \frac{1}{(p-1)^3} \ge \frac{p^2}{(p-1)^3}.$$

Case 2. $D = 4d$, where d is an odd square-free number, $d \ge 5$ (otherwise, the quadratic form with discriminant $-D$ is one-class).

Then if $(N, D) > 1$, we have $\sigma(N, D) = \sigma_0(N)$.

Suppose that $(N, D) = 1$; then

$$\sigma(N, D) \ge \frac{2}{h^3}\left(\prod_{p\mid d}\left(1 + \frac{1}{(p-1)^3}\right) - \prod_{p\mid d}\frac{p^2}{(p-1)^3}\right)\prod_{p\nmid D}\left(1 - \frac{\gamma_0(N,p)}{\varphi^3(p)}\right).$$

By assumption, d is an odd square-free number, $d \geq 5$; therefore, there exists a prime $p \geq 5$ dividing d. But, for $p \geq 5$, the following inequality holds:

$$\frac{1}{2}\left(1+\frac{1}{(p-1)^3}\right) \geq \frac{p^2}{(p-1)^3}\,,$$

whence $\sigma(N,D) \geq (1/2)\sigma_0(N)$.

Case 3. D is an odd square-free number, $D \equiv 3 \pmod 4$, $D > 15$.

We split this case into subcases.

3.1. $\chi_1(N) = 1$. Then

$$\gamma_0(N,D) = \mu(D), \qquad \chi_1(-1) = -1;$$

therefore,

$$\sigma(N,D) \geq \frac{1}{h^3}\left(\prod_{p|D}\left(1+\frac{1}{(p-1)^3}\right) - \prod_{p|D}\frac{p^2}{(p-1)^3}\prod_{p\nmid D}\left(1-\frac{\gamma_0(N,p)}{\varphi^3(p)}\right)\right) \geq \frac{1}{2}\sigma_0(N),$$

because there exists a prime divisor D greater than 5.

3.2. $\chi_1(N) = -1$. Then

$$\sigma(N,D) = \sigma_0(N) + \frac{1}{h^3}\left(\frac{-6D+D^2}{\varphi^3(D)}\right)\prod_{p\nmid D}\left(1-\frac{\gamma_0(N,p)}{\varphi^3(p)}\right) > \sigma_0(N),$$

because $-6D + D^2 > 0$.

3.3. $\chi_1(N) = 0$. In this case,

$$\sigma_1(N,D) = \sigma_3(N,D) = 0.$$

Suppose that $(N,D) = D_1 > 1,\ D = D_1D_2,\ (N,D_2) = 1$. Then $\gamma_0(N,D) = \mu(D_2)\varphi(D_1)$,

$$\sigma(N,D) = \frac{1}{h^3}\left(\prod_{p|D_1}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|D_2}\left(1+\frac{1}{(p-1)^3}\right)\right.$$
$$\left.-\,3\prod_{p|D_1}\frac{p}{(p-1)^2}\prod_{p|D_2}\frac{p}{(p-1)^3}\right)\prod_{p\nmid D}\left(1-\frac{\gamma_0(N,p)}{\varphi^3(p)}\right).$$

By assumption, $D = D_1D_2$ is an odd square-free number, $D > 15$. Therefore, either D_1 or D_2 has a prime divisor greater than 5.

If $p \mid D_1$, $p \geq 7$, then we have the inequality

$$\frac{1}{5}\left(1-\frac{1}{(p-1)^2}\right) \geq \frac{p}{(p-1)^2}\,,$$

from which it follows that, in this case, $\sigma(N,D) \geq (2/5)\sigma_0(N)$.

But if $p \mid D_2$, $p \geq 7$, then the following inequality holds:

$$\frac{1}{10}\left(1+\frac{1}{(p-1)^3}\right) \geq \frac{p}{(p-1)^3}$$

and, therefore, $\sigma(N,D) \geq (7/10)\sigma_0(N)$.

The resulting estimates imply that, for all N and D satisfying the conditions, the following inequality holds:

$$\sigma(N,D) \geq \frac{2}{5}\,\sigma_0(N) > \frac{2}{5h^3}$$

(the inequality $\sigma_0(N) > 1/h^3$ is well known; see, for example, [5, Chap. 10]). □

In the previous proposition, we did not consider the case $D = 15$. As will be seen from the next proposition, the quadratic forms with the discriminant -15 which are not one-class are the unique exception from the following general rule: there exist infinitely many odd numbers N for which $\sigma(N, 15) = 0$.

Proposition 4. *Suppose that N is an odd number. If $(N, 15) = 1$, then*

$$\sigma(N, 15) > \frac{1}{13}.$$

If $(N, 3) = 1,\ (N, 5) > 1$ or $(N, 5) = 1,\ (N, 3) > 1$, then

$$\sigma(N, 15) > \frac{1}{8}.$$

If $N \equiv 0 \pmod{15}$, then $\sigma(N, 15) = 0$; moreover, the equation

$$p_1 + p_2 + p_3 = N$$

is insoluble in primes p_1, p_2, p_3 expressible by the quadratic forms with discriminant -15.

Proof. The first three assertions immediately follow from the equality

$$\sigma(N, 15) = \frac{1}{8}\left(\prod_{p|15}\left(1 - \frac{\gamma_0(N,p)}{\varphi^3(p)}\right) + \frac{45\chi_1(N)}{512} - \frac{45\gamma_0(N,15)}{512} - \frac{225\chi_1(N)}{512}\right) \times \prod_{p\nmid 15}\left(1 - \frac{\gamma_0(N,p)}{\varphi^3(p)}\right)$$

(we have taken into account the fact that $h = 2$).

Also, any quadratic form with discriminant -15 is equivalent to one of the forms

$$f_1(x, y) = x^2 + xy + 4y^2, \qquad f_2(x, y) = 2x^2 + xy + 2y^2.$$

Simple calculations show that the forms f_1 and f_2 can represent only primes congruent to 1, 2, 4, and 8 modulo 15, while the sums of any three such numbers are not divisible by 15. $\square$

ACKNOWLEDGMENTS

This work was supported by the Ministry of Education and Science of the Russian Federation under the program "Development of the Scientific Potential of Higher School" (grant no. RNP.2.1.1.3263) and by Belgorod State University (grant no. VGK 007-04).

REFERENCES

1. Z. I. Borevich and I. R. Shafarevich, *Number Theory* (Nauka, Moscow, 1985) [in Russian].
2. S. A. Gritsenko, "On the functional equation of an arithmetic Dirichlet series," Chebyshev Sbornik **4** (2), 55–67 (2003).
3. S. A. Gritsenko, "On the distribution of the norms of simple ideals from a given class and arithmetic progressions," Zapiski Nauchn. Sem. POMI **322**, 45–62 (2005).
4. S. A. Gritsenko, "Estimate of a linear trigonometric sum over primes expressible by a given quadratic form," Chebyshev Sbornik **5** (4), 82–87 (2005).
5. A. A. Karatsuba, *Foundations of Analytic Number Theory* (Nauka, Moscow, 1983) [in Russian].