COMPACTNESS RESULT FOR PERIODIC STRUCTURES AND ITS APPLICATION TO THE HOMOGENIZATION OF A DIFFUSION-CONVECTION EQUATION

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ABSTRACT. We prove the strong compactness of the sequence $\{c^\varepsilon(\mathbf{x},t)\}$ in $L_2(\Omega_T)$, $\Omega_T=\{(\mathbf{x},t):\mathbf{x}\in\Omega\subset\mathbb{R}^3, t\in(0,T)\}$, bounded in $W_2^{1,0}(\Omega_T)$ with the sequence of time derivative $\{\partial/\partial t\big(\chi(\mathbf{x}/\varepsilon)c^\varepsilon\big)\}$ bounded in the space $L_2\big((0,T);W_2^{-1}(\Omega)\big)$. As an application we consider the homogenization of a diffusion-convection equation with a sequence of divergence-free velocities $\{\mathbf{v}^\varepsilon(\mathbf{x},t)\}$ weakly convergent in $L_2(\Omega_T)$.

1. Introduction

There are several compactness criteria and among them Tartar's method of compensated compactness [17] and the method suggested by Aubin in [6] (see also [14]). These methods intensively used in the theory of nonlinear differential equations. As a rule, the first one has applications in stationary problems, while the second method is used in non-stationary nonlinear equations.

In the present publication we discuss the method, closed to the Aubin compactness lemma. In its simplest setting, this result provides the strong compactness in $L_2(\Omega_T)$ (throughout the article, we use the customary notation of function spaces and norms [14, 13]) to the sequence of functions $\{c^\varepsilon(\mathbf{x},t)\}$ bounded in $L_\infty\big((0,T);L_2(\Omega)\big)\cap W_2^{1,0}(\Omega_T)$ with the sequence of the time derivatives $\{\partial c^\varepsilon/\partial t\}$ bounded in $L_2\big((0,T);W_2^{-1}(\Omega)\big)$. But in many applications (especially in homogenization), the second condition on a boundedness of the time derivatives in some dual space is not always satisfied. Sometimes, instead of the last condition, one has the boundedness of time derivatives in a dual space $L_2\big((0,T);W_2^{-1}(\Omega_f^\varepsilon)\big)$, defined on some periodic subdomain $\Omega_f^\varepsilon\subset\Omega$. Using new ideas of Nguetseng's two-scale convergence method [16] we prove that even under this weak condition the sequence $\{c^\varepsilon(\mathbf{x},t)\}$ still remains strongly compact in $L_2(\Omega_T)$. The main point here is the fact, that if for some $t_0\in(0,T)$,

$$\lim_{\varepsilon\to 0}\varepsilon^2\int_\Omega|\nabla c^\varepsilon(\mathbf{x},t_0)|^2\,dx=0,$$

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then the bounded in $L_2(\Omega)$ sequence $\{c^\varepsilon(\mathbf{x},t_0)\}$ contains a subsequence, which two-scale converges in $L_2(\Omega)$ to some function $\bar{c}(\mathbf{x},t_0)$.

Recall that, in general, any bounded in $L_2(\Omega)$ sequence $\{u^\varepsilon\}$ contains a two-scale convergent subsequence $\{u^{\varepsilon_k}\}$, where the limiting function $U(\mathbf{x},\mathbf{y})$ is 1-periodic in variable $\mathbf{y}\in Y=(0,1)^n$:

$$\int_\Omega u^{\varepsilon_k}(\mathbf{x})\varphi(\mathbf{x},\frac{\mathbf{x}}{\varepsilon_k})dx \to \iint_{\Omega Y} U(\mathbf{x},\mathbf{y})\varphi(\mathbf{x},\mathbf{y})dydx$$

for any smooth function $\varphi(\mathbf{x},\mathbf{y})$, 1-periodic in the variable $\mathbf{y}$. In particular, for $\varphi(\mathbf{x},\mathbf{y})=\varphi_0(\mathbf{y})\cdot h(\mathbf{x})$, where $\varphi_0\in L_2(Y)$ and $h\in L_\infty(\Omega)$.

A similar compactness result has been proved in [4] under different assumptions on the sequence $\{c^\varepsilon(\mathbf{x},t)\}$. More precisely, the corresponding [4, Lemma 4.2] states, that if for all $\varepsilon>0$

$$0\leqslant c^\varepsilon(\mathbf{x},t)\leqslant M_0, \int_{\Omega_T}|c^\varepsilon(\mathbf{x}+\triangle\mathbf{x},t)-c^\varepsilon(\mathbf{x},t)|^2\,dx\,dt\leqslant M_0\omega(|\triangle\mathbf{x}|),$$

with some $\omega(\xi)$, such that $\omega(\xi)\to 0$ as $\xi\to 0$, and

$$\|\frac{\partial}{\partial t}(\chi^\varepsilon c^\varepsilon)\|_{L_2\left((0,T);W_2^{-1}(\Omega)\right)}\leqslant M_0,$$

where $0<\chi^-\leqslant\chi^\varepsilon\leqslant\chi^+<1$, $\chi^\pm=const$, then the family $\{c^\varepsilon\}$ is a compact set in $L_2(\Omega_T)$.

As an application of our result we consider the homogenization of diffusion-convection equation

$$\frac{\partial c^\varepsilon}{\partial t}+\mathbf{v}^\varepsilon\cdot\nabla c^\varepsilon=\triangle c^\varepsilon,\quad \mathbf{x}\in\Omega^\varepsilon,\ t\in(0,T), \tag{1.1}$$

with boundary and initial conditions

$$\big(\nabla c^\varepsilon-\mathbf{v}^\varepsilon c^\varepsilon\big)\cdot\boldsymbol{\nu}=0,\quad \mathbf{x}\in\partial\Omega^\varepsilon\backslash S,\ t\in(0,T), \tag{1.2}$$

$$c^\varepsilon(\mathbf{x},t)=0,\quad \mathbf{x}\in S\cap\partial\Omega^\varepsilon,\ t\in(0,T), \tag{1.3}$$

$$c^\varepsilon(\mathbf{x},0)=c_0(\mathbf{x}),\quad \mathbf{x}\in\Omega^\varepsilon. \tag{1.4}$$

In (1.2), $\boldsymbol{\nu}$ is the unit outward normal vector to the boundary $\partial\Omega^\varepsilon$ and $S=\partial\Omega$.

We assume that velocities $\mathbf{v}^\varepsilon$ are uniformly bounded in $L_8\big((0,T);L_4(\Omega)\big)$:

$$\int_0^T\Big(\int_\Omega|\mathbf{v}^\varepsilon|^4dx\Big)^2dt\leqslant M_0^2, \tag{1.5}$$

and

$$\nabla\cdot\mathbf{v}^\varepsilon=0,\mathbf{x}\in\Omega_T. \tag{1.6}$$

As usual, the solution to the problem (1.1)–(1.4) is understood in a weak sense as a solution of the integral identity

$$\int_{\Omega_T^\varepsilon}\Big(c^\varepsilon\frac{\partial\phi}{\partial t}-\big(\nabla c^\varepsilon-\mathbf{v}^\varepsilon c^\varepsilon\big)\cdot\nabla\phi\Big)\,dx\,dt=-\int_{\Omega^\varepsilon}c_0(\mathbf{x})\phi(\mathbf{x},0)\,dx \tag{1.7}$$

for any smooth functions ϕ, such that $\phi(\mathbf{x},T)=0$.

Homogenization means the limiting procedure in (1.7) as $\varepsilon\to 0$ and the main problem here is how to pass to the limit in the nonlinear term

$$c^\varepsilon\mathbf{v}^\varepsilon\cdot\nabla\phi.$$

It has been done for velocities with a special structure

$$\mathbf{v}^{\varepsilon}=\mathbf{v}^{\varepsilon}(\mathbf{x}),\mathrm{or}\mathbf{v}^{\varepsilon}=\mathbf{v}(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon})$$

(see, for example, [5, 3, 7, 8, 9, 10]). However, in the general case we need the strong compactness in $L_2(\Omega_T)$ of the sequence $\{c^{\varepsilon}\}$. Our compactness result and the energy estimate

$$\max_{0<t<T}\int_{\Omega^{\varepsilon}}|c^{\varepsilon}(\mathbf{x},t)|^2dx+\int_{\Omega^{\varepsilon}_T}|\nabla c^{\varepsilon}(\mathbf{x},t)|^2\,dx\,dt\leqslant M_1^2$$

provide this compactness.

Note, that to apply any compactness result we must consider sequences in a fixed domain. To do that we use the well-known extension result [1] and restrict ourself with special domains Ω^{ε}:

Assumption 1.1. Let $\chi(\mathbf{y})$ be 1-periodic in the variable $\mathbf{y}$ function, such that $\chi(\mathbf{y})=1,\mathbf{y}\in Y_f\subset Y$, $\chi(\mathbf{y})=0,\mathbf{y}\in Y_s=Y\backslash\overline{Y}_f$.

(1) The set Y_f is an open one and $\gamma=\partial Y_f\cap\partial Y_s$ is a Lipschitz continuous surface.
(2) Let Y^{ε}_f be a periodic repetition in $\mathbb{R}^n$ of the elementary cell εY_f. Then Y^{ε}_f is a connected set with a Lipschitz continuous boundary $\partial Y^{\varepsilon}_f$.
(3) $\Omega\subset\mathbb{R}^n$ is a bounded domain with a Lipschitz continuous boundary $S=\partial\Omega$ and $\Omega^{\varepsilon}=\Omega\cap Y^{\varepsilon}_f$.

Due to periodicity of Y^{ε}_f the characteristic function of the domain Ω^{ε} in Ω has a form:

$$\chi^{\varepsilon}(\mathbf{x})=\chi(\frac{\mathbf{x}}{\varepsilon}).$$

For such domains Ω^{ε} the extension theorem [1] allows us to construct a linear operator $\mathbb{A}^{\varepsilon}$

$$\mathbb{A}^{\varepsilon}:W^1_2(\Omega^{\varepsilon})\to W^1_2(\Omega),\tilde{c}^{\varepsilon}=\mathbb{A}^{\varepsilon}(c^{\varepsilon}),\tag{1.8}$$

such that

$$\int_{\Omega}|\tilde{c}^{\varepsilon}(\mathbf{x},t)|^2dx\leqslant C_0\int_{\Omega^{\varepsilon}}|c^{\varepsilon}(\mathbf{x},t)|^2dx,\tag{1.9}$$

$$\int_{\Omega}|\nabla\tilde{c}^{\varepsilon}(\mathbf{x},t)|^2dx\leqslant C_0\int_{\Omega^{\varepsilon}}|\nabla c^{\varepsilon}(\mathbf{x},t)|^2dx.\tag{1.10}$$

where the constant $C_0=C_0(\Omega,Y_f)$ does not depend on ε and $t\in(0,T)$.

2. Main results

Our principal result is the following

Theorem 2.1. *Let $\{\tilde{c}^{\varepsilon}(\mathbf{x},t)\}$ be a bounded sequence in $L_\infty\big((0,T);L_2(\Omega)\big)\cap W^{1,0}_2(\Omega_T)$ and weakly convergent in $L_2\big((0,T);L_2(\Omega)\big)\cap W^{1,0}_2(\Omega_T)$ to a function $c(\mathbf{x},t)$. Also let the sequence $\{\partial/\partial t(\chi^{\varepsilon}(\mathbf{x})\tilde{c}^{\varepsilon}(\mathbf{x},t))\}$ be bounded in $L_2\big((0,T);W^{-1}_2(\Omega)\big)$, where $\chi^{\varepsilon}(\mathbf{x})=\chi(\mathbf{x}/\varepsilon)$, $\chi(\mathbf{y})$ is 1-periodic in the variable $\mathbf{y}$ measurable bounded function, such that*

$$\langle\chi\rangle_Y=\int_Y\chi(\mathbf{y})dy=m\neq0,$$

and Y is the unit cube in $\mathbb{R}^n$. Then the sequence $\{\tilde{c}^{\varepsilon}(\mathbf{x},t)\}$ converges strongly in $L_2(\Omega_T)$ to its weak limit $c(\mathbf{x},t)$.

As an application of this result we consider a homogenization of the problem (1.1)–(1.4).

We prove the following result.

Theorem 2.2. *Under conditions* (1.5)–(1.6) *and Assumption 1.1 let* $c^{\varepsilon}(\mathbf{x},t)$ *be the solution to the problem* (1.1)–(1.4), $c_0 \in L_2(\Omega)$,

$$\int_{\Omega} |c_0|^2 dx \leqslant M_0^2, \tag{2.1}$$

and

$$\tilde{\mathbf{v}}^{\varepsilon} \rightharpoonup \mathbf{v} \quad \text{weakly in } L_2(\Omega_T), \tag{2.2}$$

where $\tilde{\mathbf{v}}^{\varepsilon}(\mathbf{x},t) = \chi^{\varepsilon}(\mathbf{x})\mathbf{v}^{\varepsilon}(\mathbf{x},t)$. *Then the sequence* $\{\tilde{c}^{\varepsilon}\}$, *where* $\tilde{c}^{\varepsilon} = \mathbb{A}^{\varepsilon}(c^{\varepsilon})$, *converges strongly in* $L_2(\Omega_T)$ *and weakly in* $W_2^{1,0}(\Omega_T)$ *to the solution* $c(\mathbf{x},t)$ *of the homogenized equation*

$$m\frac{\partial c}{\partial t} = \nabla \cdot \big(\mathbb{B}\cdot\nabla c + (\mathbf{v}_0 - \mathbf{v})c\big), \quad \mathbf{x} \in \Omega, t \in (0,T), \tag{2.3}$$

with boundary and initial conditions

$$c(\mathbf{x},t) = 0, \mathbf{x} \in S, t \in (0,T), \tag{2.4}$$

$$c(\mathbf{x},0) = c_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.5}$$

In (2.3) *the symmetric strictly positively defined constant matrix* $\mathbb{B}$ *and the vector* $\mathbf{v}_0$ *are given below by formulas* (4.13) *and* (4.14).

3. Proof of Theorem 2.1

We split the proof into several independent steps. As a first step we prove the following.

Lemma 3.1. *Under conditions of Theorem 2.1 the sequence* $\{\chi^{\varepsilon}(\mathbf{x})\tilde{c}^{\varepsilon}(\mathbf{x},t)\}$ *converges weakly in* $L_2(\Omega)$ *to the function* $mc(\mathbf{x},t)$ *for almost all* $t \in (0,T)$.

Proof. By the properties of the two-scale convergence [16, 15] the sequence $\{\tilde{c}^{\varepsilon}\}$ two-scale converges in $L_2(\Omega_T)$ to the function $c(\mathbf{x},t)$. That is, for any 1-periodic in variable $\mathbf{y}$ smooth function $\varphi(\mathbf{x},\mathbf{y},t)$

$$\int_{\Omega_T} \tilde{c}^{\varepsilon}(\mathbf{x},t)\varphi(\mathbf{x},\frac{\mathbf{x}}{\varepsilon},t)\,dx\,dt \to \int_{\Omega_T} c(\mathbf{x},t)\Big(\int_Y \varphi(\mathbf{x},\mathbf{y},t)dy\Big)dxdt.$$

In particular, this relation holds true for $\varphi = \varphi_0(\mathbf{x},t)\varphi_1(\mathbf{y})$ with $\varphi_0 \in L_\infty(\Omega_T)$ and $\varphi_1 \in L_2(Y)$. If we choose

$$\varphi(\mathbf{x},\frac{\mathbf{x}}{\varepsilon},t) = \chi(\frac{\mathbf{x}}{\varepsilon})\eta(t)\psi(\mathbf{x}) = \chi^{\varepsilon}(\mathbf{x})\eta(t)\psi(\mathbf{x}),$$

then

$$\int_{\Omega_T} \tilde{c}^{\varepsilon}(\mathbf{x},t)\chi^{\varepsilon}(\mathbf{x})\eta(t)\psi(\mathbf{x})\,dx\,dt \to \int_{\Omega_T} mc(\mathbf{x},t)\eta(t)\psi(\mathbf{x})\,dx\,dt. \tag{3.1}$$

Let

$$f_{\psi}^{\varepsilon}(t) = \int_{\Omega} \chi^{\varepsilon}(\mathbf{x})\tilde{c}^{\varepsilon}(\mathbf{x},t)\psi(\mathbf{x})dx, \quad f_{\psi}(t) = \int_{\Omega} mc(\mathbf{x},t)\psi(\mathbf{x})dx.$$

Then the above relation means that

$$\int_0^T \eta(t)f_{\psi}^{\varepsilon}(t)dt \to \int_0^T \eta(t)f_{\psi}(t)dt, \tag{3.2}$$

for any functions $\eta \in L_\infty(0,T)$ and $\psi \in L_\infty(\Omega)$.

To prove the lemma we have to show that for almost all $t \in (0,T)$ functions $f^\varepsilon_\psi(t)$ pointwise converge to the function $f_\psi(t)$. First of all, we restrict ourself with functions $\psi \in \overset{\circ}{W}{}^1_2(\Omega)$.

By the assumptions in Theorem 2.1, the time derivatives $\partial/\partial t\big(\chi^\varepsilon(\mathbf{x})\tilde{c}^\varepsilon\big)$ belong to the space $L_2\big((0,T);\overset{\circ}{W}{}^{-1}_2(\Omega)\big)$ and uniformly bounded there. This means that there exists a sequence $\{\mathbf{F}^\varepsilon(\mathbf{x},t)\}$, such that

$$\int_{\Omega_T} |\mathbf{F}^\varepsilon|^2\, dx\, dt \leqslant M_0^2,$$

and

$$\int_{\Omega_T} \frac{d\varphi(t)}{dt}\chi^\varepsilon(\mathbf{x})\tilde{c}^\varepsilon(\mathbf{x},t)\psi(\mathbf{x})\, dx\, dt = \int_{\Omega_T} \varphi(t)\mathbf{F}^\varepsilon(\mathbf{x},t)\cdot\nabla\psi(\mathbf{x})\, dx\, dt \tag{3.3}$$

for any $\varphi \in^1_2 (0,T)$ and $\psi \in \overset{\circ}{W}{}^1_2(\Omega)$. If we put

$$g^\varepsilon(t) = -\int_\Omega \mathbf{F}^\varepsilon(\mathbf{x},t)\cdot\nabla\psi(\mathbf{x})d\mathbf{x},$$

then

$$\int_0^T |g^\varepsilon|^2 dt \leqslant M_0^2\|\nabla\psi\|^2_{2,\Omega} = M_\psi^2,$$

and identity (3.3) rewrites as

$$\int_0^T \big(f^\varepsilon_\psi(t)\frac{d\varphi(t)}{dt} + \varphi(t)g^\varepsilon(t)\big)dt = 0. \tag{3.4}$$

Therefore by [2], the function $f^\varepsilon_\psi(t)$ possesses the generalized time derivative $g^\varepsilon(t) \in L_2(0,T)$ and takes place a representation

$$f^\varepsilon_\psi(t) = f^\varepsilon_\psi(t_\varepsilon) + \int_{t_\varepsilon}^t g^\varepsilon(\tau)d\tau, |f^\varepsilon_\psi(t_\varepsilon)| \leqslant M_\psi.$$

In particular,

$$|f^\varepsilon_\psi(t)| \leqslant M_\psi, |f^\varepsilon_\psi(t_1) - f^\varepsilon_\psi(t_2)| \leqslant M_\psi|t_2 - t_1|^{1/2}. \tag{3.5}$$

Thus, we may apply the Ascoli-Arzela theorem [12] and state that there exists some subsequence $\{\varepsilon_m\}$, such that the sequence of continuous functions $\{f^{\varepsilon_m}_\psi(t)\}$ uniformly converges to some continuous function $\overline{f}_\psi(t)$:

$$f^{\varepsilon_m}_\psi(t) \rightrightarrows \overline{f}_\psi(t), \quad \text{as } \varepsilon_m \to 0, \forall t \in (0,T). \tag{3.6}$$

Therefore,

$$\int_0^T \eta(t)f^{\varepsilon_m}_\psi(t)dt \to \int_0^T \eta(t)\overline{f}_\psi(t)dt, \quad \text{as } \varepsilon_m \to 0. \tag{3.7}$$

But, on the other hand, according to (3.1)

$$\int_0^T \eta(t)f^{\varepsilon_m}_\psi(t)dt \to \int_0^T \eta(t)f_\psi(t)dt, \text{as} \varepsilon_m \to 0. \tag{3.8}$$

By the arbitrary choice of $\eta(t)$ (3.6)-(3.8) result

$$f^{\varepsilon_m}_\psi(t) \to f_\psi(t) \quad \text{as } \varepsilon_m \to 0, \text{ for almost all } t \in [0,T].$$

Due to the uniqueness of the limit, the last relation holds for the entire sequence $\{f^{\varepsilon}_{\psi}(t)\}$:

$$f^{\varepsilon}_{\psi}(t) = \int_{\Omega} \chi^{\varepsilon}(\mathbf{x}) c^{\varepsilon}(\mathbf{x},t)\psi(\mathbf{x})dx \to \int_{\Omega} mc(\mathbf{x},t)\psi(\mathbf{x})dx = f_{\psi}(t)$$

as $\varepsilon \to 0$ for almost all $t \in (0,T)$. □

As a next step we prove the following result.

Lemma 3.2. *Under conditions of Theorem 2.1 there exists a subsequence $\{\varepsilon_k\}$, such that*

$$\lim_{\varepsilon_k \to 0} \varepsilon_k^2 \int_{\Omega} |\nabla \tilde{c}^{\varepsilon_k}(\mathbf{x},t_0)|^2 dx = 0 \tag{3.9}$$

for almost all $t_0 \in (0,T)$.

Proof. In fact, the boundedness of the sequence $\{\nabla \tilde{c}^{\varepsilon}\}$ in $L_2(\Omega_T)$ implies

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_{\Omega_T} |\nabla \tilde{c}^{\varepsilon}(\mathbf{x},t)|^2 \, dx\, dt = 0. \tag{3.10}$$

Let

$$u^{\varepsilon}(t) = \varepsilon^2 \int_{\Omega} |\nabla \tilde{c}^{\varepsilon}(\mathbf{x},t)|^2 dx.$$

Then the relation (3.10) means that the sequence $\{u^{\varepsilon}\}$ converges to zero in $L_1(0,T)$. Due to the well-known theorem of functional analysis [12] there exists some subsequence $\{\varepsilon_k\}$, such that the sequence $\{u^{\varepsilon_k}(t_0)\}$ pointwise converge to zero for almost all $t_0 \in (0,T)$:

$$u^{\varepsilon_k}(t_0) \to 0 \quad \text{for almost all } t_0 \in (0,T).$$

The above relation proves (3.9). □

The following statement is a crucial one and essentially uses the notion of two-scale convergence.

Lemma 3.3. *Under the conditions of Theorem 2.1, the sequence $\{\tilde{c}^{\varepsilon_k}(\mathbf{x},t_0)\}$ two-scale converges in $L_2(\Omega)$ to the function $c(\mathbf{x},t_0)$ for almost all $t_0 \in (0,T)$.*

Proof. Let $Q \subset (0,T)$ be the set of full measure in $(0,T)$, where hold true conditions of the Lemma 3.1 and condition (3.9).

By hypothesis, the sequence $\{\tilde{c}^{\varepsilon_k}(\mathbf{x},t_0)\}$ for $t_0 \in Q$ is bounded in $L_2(\Omega)$. Therefore, there exists some subsequence which two-scale converges in $L_2(\Omega)$ to some 1-periodic in variable $\mathbf{y}$ function $\overline{C}(\mathbf{x},\mathbf{y},t_0) \in L_2(\Omega \times Y)$. Applying integration by parts

$$\begin{aligned}
&\varepsilon_k \int_{\Omega} \nabla c^{\varepsilon_k}(\mathbf{x},t_0) \cdot \boldsymbol{\varphi}(\frac{\mathbf{x}}{\varepsilon_k})\psi(\mathbf{x})dx \\
&= -\varepsilon_k \int_{\Omega} c^{\varepsilon_k}(\mathbf{x},t_0)\boldsymbol{\varphi}(\frac{\mathbf{x}}{\varepsilon_k}) \cdot \nabla\psi(\mathbf{x})dx - \int_{\Omega} c^{\varepsilon_k}(\mathbf{x},t_0)\big(\nabla_y \cdot \boldsymbol{\varphi}(\frac{\mathbf{x}}{\varepsilon_k})\big)\psi(\mathbf{x})dx
\end{aligned}$$

for arbitrary functions $\boldsymbol{\varphi} \in W^1_2(Y)$ and $\psi \in \overset{\circ}{W}{}^1_2(\Omega)$, and relation (3.9) we arrive at the equality

$$\int_{\Omega} \psi(\mathbf{x})\Big(\int_{Y} \overline{C}(\mathbf{x},\mathbf{y},t_0)\nabla_y \cdot \boldsymbol{\varphi}(\mathbf{y})dy\Big)dx = 0 \tag{3.11}$$

after passing to the limit as $\varepsilon_k \to 0$.

By the arbitrary choice of test functions φ and ψ, the last integral identity implies

$$\overline{C}(\mathbf{x},\mathbf{y},t_0)=\overline{c}(\mathbf{x},t_0). \tag{3.12}$$

Thus, the chosen subsequence of the sequence $\{c^{\varepsilon_k}(\mathbf{x},t_0)\}$ two-scale converges in $L_2(\Omega)$ to the function $\overline{c}(\mathbf{x},t_0)$. In particular, by the properties of two-scale convergent sequences [16] the same subsequence of $\{\chi^{\varepsilon_k}(\mathbf{x})c^{\varepsilon_k}(\mathbf{x},t_0)\}$, where $\chi^{\varepsilon_k}(\mathbf{x})=\chi(\mathbf{x}/\varepsilon_k)$, weakly converges in $L_2(\Omega)$ to the function $m\overline{c}(\mathbf{x},t_0)$. On the other hand, due to Lemma 3.1 this subsequence weakly converges in $L_2(\Omega)$ to the function $mc(\mathbf{x},t_0)$. The uniqueness of the weak limit results the equality

$$\overline{c}(\mathbf{x},t_0)=c(\mathbf{x},t_0)$$

and the convergence of the entire sequence $\{c^{\varepsilon_k}(\mathbf{x},t_0)\}$ to the same limit. □

Lemma 3.4. *Under the conditions of Theorem 2.1, the sequence $\{\tilde{c}^{\varepsilon_k}\}$ converges strongly in $L_2(\Omega_T)$ to the function $c(\mathbf{x},t)$.*

Proof. Let

$$\mathbb{H}^1=W_2^1(\Omega)\subset\mathbb{H}^0=L_2(\Omega)\subset\mathbb{H}^{-1}=W_2^{-1}(\Omega).$$

It is well known that $\mathbb{H}^1$ is compactly imbedded in $\mathbb{H}^0$, and $\mathbb{H}^0$ is compactly imbedded in $\mathbb{H}^{-1}$ ([14], [2]). The first imbedding provides for any $\eta>0$ an existence of some constant C_η such that

$$\|\tilde{c}^{\varepsilon_k}-c\|_{\mathbb{H}^0}(t)\leqslant\eta\|\tilde{c}^{\varepsilon_k}-c\|_{\mathbb{H}^1}(t)+C_\eta\|\tilde{c}^{\varepsilon_k}-c\|_{\mathbb{H}^{-1}}(t)$$

for all k and for all $t\in[0,T]$ (see [14]). Therefore,

$$\int_0^T\|\tilde{c}^{\varepsilon_k}-c\|^2_{\mathbb{H}^0}(t)dt\leqslant\eta\int_0^T\|\tilde{c}^{\varepsilon_k}-c\|^2_{\mathbb{H}^1}(t)dt+C_\eta\int_0^T\|\tilde{c}^{\varepsilon_k}-c\|^2_{\mathbb{H}^{-1}}(t)dt$$
$$\leqslant 2\eta M_0^2+C_\eta\int_0^T\|\tilde{c}^{\varepsilon_k}-c\|^2_{\mathbb{H}^{-1}}(t)dt.$$

Due to the compact imbedding $\mathbb{H}^0\to\mathbb{H}^{-1}$, the weak convergence in $\mathbb{H}^0$ of the sequence $\{\tilde{c}^{\varepsilon_k}(\mathbf{x},t_0)\}$ to the function $c(\mathbf{x},t_0)$ for all $t_0\in Q$, and the dominated convergence theorem [12] one has

$$\int_0^T\|\tilde{c}^{\varepsilon_k}-c\|^2_{\mathbb{H}^{-1}}(t)dt\to 0\quad\text{as } k\to\infty.$$

This last fact and the arbitrary choice of the constant η prove the statement of the lemma. □

4. Proof of Theorem 2.2

To simplify the proof we additionally suppose that

Assumption 4.1. (1) $Y_s\subset Y,\gamma\cap\partial Y=\emptyset$;
(2) the domain Ω is a unit cube;
(3) $1/\varepsilon$ is an integer.

As before, we divide the proof by several steps. As a first step we state the well-known existence and uniqueness result for solutions of the problem (1.1)–(1.3) (see [13]).

Lemma 4.2. *Under conditions of Theorem 2.2 for all* $\varepsilon > 0$ *the problem* (1.1)–(1.4) *has a unique solution*

$$c^\varepsilon \in L_\infty\big((0,T); L_2(\Omega^\varepsilon)\big) \cap W_2^{1,0}(\Omega_T^\varepsilon)$$

and

$$\max_{0<t<T} \int_{\Omega^\varepsilon} |c^\varepsilon(\mathbf{x},t)|^2 dx + \int_{\Omega_T^\varepsilon} |\nabla c^\varepsilon|^2 \, dx\, dt \leqslant M_1^2. \tag{4.1}$$

To get the basic estimate (4.1) we first rewrite (1.1) in the form

$$\frac{\partial c^\varepsilon}{\partial t} = \nabla \cdot (\nabla c^\varepsilon - \mathbf{v}^\varepsilon c^\varepsilon),$$

multiply by c^ε and integrate by parts over domain Ω^ε:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\varepsilon} |c^\varepsilon(\mathbf{x},t)|^2 dx + \int_{\Omega^\varepsilon} |\nabla c^\varepsilon|^2 dx = \int_{\Omega^\varepsilon} c^\varepsilon \mathbf{v}^\varepsilon \cdot \nabla c^\varepsilon dx.$$

Let $\tilde{c}^\varepsilon(.,t) = \mathbb{A}^\varepsilon\big(c^\varepsilon(.,t)\big)$ be an extension of the function c^ε onto domain Ω. Then

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} \chi^\varepsilon |\tilde{c}^\varepsilon(\mathbf{x},t)|^2 dx + \int_{\Omega} \chi^\varepsilon |\nabla \tilde{c}^\varepsilon|^2 dx = \int_{\Omega} \chi^\varepsilon \tilde{c}^\varepsilon \mathbf{v}^\varepsilon \cdot \nabla \tilde{c}^\varepsilon dx \equiv J_1. \tag{4.2}$$

To estimate J_1 we use the Hölder inequality:

$$\begin{aligned} |J_1| &\leqslant \Big(\int_\Omega \chi^\varepsilon |\mathbf{v}^\varepsilon|^4 dx\Big)^{1/4} \cdot \Big(\int_\Omega \chi^\varepsilon |\tilde{c}^\varepsilon|^4 dx\Big)^{1/4} \cdot \Big(\int_\Omega \chi^\varepsilon |\nabla \tilde{c}^\varepsilon|^2 dx\Big)^{1/2} \\ &\leqslant \Big(\int_\Omega \chi^\varepsilon |\mathbf{v}^\varepsilon|^4 dx\Big)^{1/4} \cdot \Big(\int_\Omega |\tilde{c}^\varepsilon|^4 dx\Big)^{1/4} \cdot \Big(\int_\Omega |\nabla \tilde{c}^\varepsilon|^2 dx\Big)^{1/2}. \end{aligned}$$

Due to Assumption 4.1

$$\tilde{c}^\varepsilon \in \overset{\circ}{W}{}_2^1(\Omega),$$

and we may apply the well-known interpolation inequality (see [13])

$$\Big(\int_\Omega |\tilde{c}^\varepsilon|^4 dx\Big)^{1/4} \leqslant \beta \Big(\int_\Omega |\tilde{c}^\varepsilon|^2 dx\Big)^{1/8} \cdot \Big(\int_\Omega |\nabla \tilde{c}^\varepsilon|^2 dx\Big)^{3/8}.$$

Therefore (see (1.9) and (1.10))

$$\begin{aligned} |J_1| &\leqslant \beta\Big(\int_\Omega \chi^\varepsilon |\mathbf{v}^\varepsilon|^4 dx\Big)^{1/4} \cdot \Big(\int_\Omega |\tilde{c}^\varepsilon|^2 dx\Big)^{1/8} \cdot \Big(\int_\Omega |\nabla \tilde{c}^\varepsilon|^2 dx\Big)^{7/8} \\ &\leqslant C_0\beta\Big(\int_\Omega \chi^\varepsilon |\mathbf{v}^\varepsilon|^4 dx\Big)^{1/4} \cdot \Big(\int_\Omega \chi^\varepsilon |\tilde{c}^\varepsilon|^2 dx\Big)^{1/8} \cdot \Big(\int_\Omega \chi^\varepsilon |\nabla \tilde{c}^\varepsilon|^2 dx\Big)^{7/8}. \end{aligned}$$

Applying Young's and Gronwall inequalities and using assumption (1.5) and properties of the extension operator $\mathbb{A}^\varepsilon$ we arrive at

$$\max_{0<t<T} \int_{\Omega} |\tilde{c}^\varepsilon(\mathbf{x},t)|^2 dx + \int_{\Omega_T} |\nabla \tilde{c}^\varepsilon|^2 \, dx\, dt \leqslant M_1^2, \tag{4.3}$$

which is obviously equivalent to (4.1).

The integral identity for the function $\tilde{c}^\varepsilon$ with test functions $\phi = \varphi(t)\psi(\mathbf{x})$, $\varphi \in \overset{\circ}{W}{}_2^1(0,T)$, $\psi \in \overset{\circ}{W}{}_2^1(\Omega)$ takes a form

$$\int_{\Omega_T} \frac{d\varphi}{dt}(t)\chi^\varepsilon \tilde{c}^\varepsilon \psi(\mathbf{x})\, dx\, dt = \int_{\Omega_T} \varphi(t)\chi^\varepsilon\big(\nabla \tilde{c}^\varepsilon - \mathbf{v}^\varepsilon \tilde{c}^\varepsilon\big) \cdot \nabla\psi(\mathbf{x})\, dx\, dt.$$

Thus,

$$\frac{\partial}{\partial t}\big(\chi^\varepsilon(\mathbf{x})\tilde{c}^\varepsilon\big) \in L_2\big((0,T); W_2^{-1}(\Omega)\big),$$

and we may apply Theorem 2.1 and Nguetseng's Theorem [16] to state, that up to some subsequence the sequence $\{\tilde{c}^\varepsilon\}$ weakly in $\overset{\circ}{W}{}_2^{1,0}(\Omega_T)$ and strongly in $L_2(\Omega_T)$ converges to the function $c(\mathbf{x},t)$, and the sequence $\{\nabla\tilde{c}^\varepsilon\}$ two-scale converges in $L_2(\Omega_T)$ to 1-periodic in variable $\mathbf{y}$ function $\nabla c(\mathbf{x},t)+\nabla_y C(\mathbf{x},\mathbf{y},t)$.

We may also assume that the sequence $\{\mathbf{v}^\varepsilon\}$ two-scale converges to 1-periodic in variable $\mathbf{y}$ function $\mathbf{V}(\mathbf{x},\mathbf{y},t)$.

The next lemmas are standard. We derive the macro-and microscopic equations and find the solution of microscopic equation.

Lemma 4.3. *Under conditions of Theorem 2.2, the two-scale limits $c(\mathbf{x},t)$ and $C(\mathbf{x},\mathbf{y},t)$ satisfy the macroscopic integral identity*

$$\int_{\Omega_T}\Big(mc\frac{\partial\phi}{\partial t}-\big(m\nabla c+\langle\nabla_y C\rangle_{Y_f}-\mathbf{v}c\big)\cdot\nabla\phi\Big)\,dx\,dt=-\int_\Omega mc_0(\mathbf{x})\phi(\mathbf{x},0)dx \quad (4.4)$$

for arbitrary smooth functions $\phi(\mathbf{x},t)$, such that $\phi(\mathbf{x},T)=0$, which is equivalent to the macroscopic equation

$$m\frac{\partial c}{\partial t}=\nabla\cdot\big(m\nabla c+\langle\nabla_y C\rangle_{Y_f}-c\mathbf{v}\big),\quad \mathbf{x}\in\Omega, t\in(0,T), \quad (4.5)$$

with boundary and initial conditions

$$c(\mathbf{x},t)=0,\mathbf{x}\in S, t\in(0,T), \quad (4.6)$$

$$c(\mathbf{x},0)=c_0(\mathbf{x}),\quad \mathbf{x}\in\Omega. \quad (4.7)$$

To prove this lemma we just fulfill the two-scale limit as $\varepsilon\to 0$ in the integral identity (1.7) for the functions $\tilde{c}^\varepsilon$ in the form

$$\int_{\Omega_T}\chi^\varepsilon\Big(\tilde{c}^\varepsilon\frac{\partial\phi}{\partial t}-\big(\nabla\tilde{c}^\varepsilon-\tilde{\mathbf{v}}^\varepsilon\tilde{c}^\varepsilon\big)\cdot\nabla\phi\Big)\,dx\,dt=-\int_\Omega\chi^\varepsilon c_0(\mathbf{x})\phi(\mathbf{x},0)dx \quad (4.8)$$

with the test functions $\phi=\phi(\mathbf{x},t)$.

Lemma 4.4. *Under conditions of Theorem 2.2 the two-scale limits $c(\mathbf{x},t)$ and $C(\mathbf{x},\mathbf{y},t)$ satisfy the microscopic integral identity*

$$\int_Y\chi(\mathbf{y})\big(\nabla c+\nabla_y C-c\mathbf{V}\big)\cdot\nabla\phi_1\,dy=0 \quad (4.9)$$

for arbitrary 1-periodic in variable $\mathbf{y}$ smooth functions $\phi_1(\mathbf{y})$.

The integral identity (4.9) follows from (4.8) after fulfilling the two-scale limit as $\varepsilon\to 0$ with test functions $\phi=\varepsilon\phi_0(\mathbf{x},t)\phi_1(\mathbf{x}/\varepsilon)$.

Lemma 4.5. *Let $C^{(i)}(\mathbf{y}), i=1,2,3$, be the solution to the integral identity*

$$\int_Y\chi(\mathbf{y})\big(\mathbf{e}_i+\nabla_y C^{(i)}\big)\cdot\nabla\phi_1\,dy=0, \quad (4.10)$$

and $C^{(0)}(\mathbf{y},\mathbf{x},t)$ be the solution to the integral identity

$$\int_Y\chi(\mathbf{y})\big(\mathbf{V}+\nabla_y C^{(0)}\big)\cdot\nabla\phi_1\,dy=0, \quad (4.11)$$

with arbitrary 1-periodic in variable $\mathbf{y}$ smooth functions $\phi_1(\mathbf{y})$. Then the function

$$C(\mathbf{x},\mathbf{y},t)=\big(\sum_{i=1}^{3}C^{(i)}(\mathbf{y})\otimes\mathbf{e}_i\big)\cdot\nabla c(\mathbf{x},t)+C^{(0)}(\mathbf{y},\mathbf{x},t)c(\mathbf{x},t) \quad (4.12)$$

solves the integral identity (4.9).

In (4.10)–(4.12) $\mathbf{e}_i$ *is the standard Cartesian basis vector and the matrix* $\mathbf{a}\otimes\mathbf{b}$ *is defined by the formula*

$$(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c}=\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$$

for any vectors $\mathbf{a},\mathbf{b},\mathbf{c}$.

The proof of the lemma is straightforward. It is omitted.

Substitution (4.12) into (4.5) gives us desired homogenized equation (2.3) with boundary and initial conditions (2.4)–(2.5).

The matrix $\mathbb{B}$ and the vector $\mathbf{v}_0(\mathbf{x},t)$ are defined as

$$\mathbb{B}=m\mathbb{I}+\Big(\sum_{i=1}^{3}\langle\nabla_y C^{(i)}\rangle_{Y_f}\otimes\mathbf{e}_i\Big), \tag{4.13}$$

$$\mathbf{v}_0(\mathbf{x},t)=\langle\nabla_y C^{(0)}\rangle_{Y_f}, \tag{4.14}$$

where by definition $\langle f\rangle_{Y_f}=\int_{Y_f}f(\mathbf{y})dy$.

Lemma 4.6. *The matrix* $\mathbb{B}$ *is symmetric and strictly positively defined.*

The proof is well-known, see [7, 11].

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