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SYSTEMS ANALYSIS

HOPF BIFURCATION IN COASTAL ECOGEOSYSTEMS

I. T. Selezov,[1] **Iu. G. Kryvonos,**[2] **and V. M. Moskovkin**[3]

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Abstract. *The qualitative theory of differential equations (catastrophe theory) is used to analyze the optimal preservation of the ecological state of an object. As the object, we consider a seacoast that is subject to permanent destruction by waves and is fed to maintain its initial state. We model the maintenance of the equilibrium state of ecological system in some optimal mode. Unlike the well-known approaches, such model takes into account essentially nonlinear effects and control by means of beach feed, which can be interpreted as feedback. The analysis is carried out by the methods from stability theory. Characteristics of the limit cycle are obtained and its stability is analyzed.*

Keywords: *Hopf bifurcation, ecogeosystem, stability, catastrophe theory.*

INTRODUCTION

Finding optimal decisions to preserve ecological equilibrium of structures is a complicated multiparameter problem. In actual practice, ecological equilibrium is unstable; moreover, it worsens.

Any beach, natural or artificial, is supposed to keep the equilibrium if it is considered for the time interval of several years. In geological time, coastal zone cannot remain invariable. Some of them are never in equilibrium state. Equilibrium coast is such that do not vary noticeably within one or two decades.

Breakwaters are intended not only to soften the impact of storm waves on coast, but also to "retain" beach pebble on a certain area. Beaches need correct maintenance and annual supplement since 5 to 12% of pebble per year are abraded and washed off by waves in spite of the fact that solid volcanic rock is put on the shore. If it is not supplemented regularly, then water will come close to coastal retaining walls and waves, splitting upon them, will considerably deepen the bottom. It will be impossible to reconstruct such a beach.

The problem of rearrangement of sea near-shore under abrasion and sedimentation of deposits has been important for a long time [1–10].

A determining feature for generation of coastal current and transport of deposits, and hence of coast erosion, is the beginning of wave breaking and successive damping of wave heights propagating through inshore. For example, it is shown in [11, 12] that surging breaker and field of turbulence velocities have dominating influence on processes in coastal zone such as wave damping, flows of mass and impulse, currents, and sediment transport. On small water, breakers interact with seabed plants and sediments.

The coast evolution forecast was also analyzed in [13]. In case of frontal wave run-up (cliff), the equilibrium profile varies substantially [14].

[1]Institute of Hydromechanics, National Academy of Sciences of Ukraine, Kyiv, Ukraine, *selezov@ya.ru*. [2]V. M. Glushkov Institute of Cybernetics, National Academy of Sciences of Ukraine, Kyiv, Ukraine, *aik@public.icyb.kiev.ua.* [3]V. N. Karazin Kharkiv National University, Kharkiv, Ukraine, *moskovkin@bsu.edu.ru*. Translated from Kibernetika i Sistemnyi Analiz, No. 4, July–August, 2016, pp. 55–64. Original article submitted February 1, 2015.

In the present paper, we will use the qualitative theory of differential equations to analyze the ecological equilibrium control of coast subject to sustainable influence of sea waves leading to their slow or fast fracture. We will apply the integral model in the form of strongly nonlinear coupled system of two differential equations, which takes into account the volume of fragmentary material, height of coastal cliff, material's inflow and ablation, and transport of sediments. We will analyze singular points, bifurcation set, Lyapunov stability criteria, and limit cycle characteristics. (The corresponding report was presented at Lyapunov's readings in Kharkov [15].)

The approach developed in the present study is applicable for the analysis of "back–forward" scenario. If the previous ecological state, for example 50–100 years ago, is known as well as today's state, then it is possible to estimate its evolution proceeding from the estimate of oscillation period for specific parameters.

PROBLEM STATEMENT

Consider linear equation of balance of fragmentary material in coastal zone of an abrasion shore [16], where biogenic production of this material is additionally taken into account and the coefficient of its abradability due to wave impact linearly depends on the biomass of seabed biocenose:

$$\frac{dW}{dt} = aH\gamma\,(W_m - W) - [C_0\,(1 - B / B_{\max} + C_{\min})]\,W + U + \delta B, \tag{1}$$

$$\frac{dB}{dt} = k_1 B(1 - B / B_{\max}) - k_2 W, \tag{2}$$

where $k_1, k_2 = \text{const} > 0$.

The following notation is accepted in formulas (1) and (2): t is time, year; W is the volume of fragmentary material per unit of shore line length, m^3/m ($0 \le W \le W_m$); W_m is limiting (minimum) volume of fragmentary material at the beach at which abrasion stops; a is the share of fragmentary beach-forming material (not suspension-forming fractions) in the rocks that compose the coast ($0 \le a \le 1$); H is the height of coastal cliff, m; $\gamma(W_m - W)$ is the velocity of coastal cliff retreat, m/year; $\gamma = \text{const} > 0.1$, $\text{m}^{-1}\text{year}^{-1}$; B is the biomass of seabed biocenosis per unit width of abrasion shoal, ton/m ($0 \le B \le B_{\max}$); U is the intensity of inflow ($U > 0$) or ablation ($U < 0$) of the material due to natural (transport of deposits by currents) or artificial (filling, withdrawal) factors, m^2/year; δ is the coefficient of biogenic production of fragmentary material, m^3/(ton per year) (amount of the material obtained from one ton of the zoobenthos biomass per year), $C_0, C_{\min} = \text{const} > 0$, year^{-1}; $k = C_0(1 - B / B_{\max}) + C_{\min}$ is the coefficient of abradability of the material, year^{-1} (linear approximation between its two characteristic values: $k(B = 0) = k_{\max} = C_0 + C_{\min}$ and $k(B = B_{\max}) = k_{\min} = C_{\min}$.

Equation (2) is based on ecological and lithodynamic features of the process under study: self-adaptive growth of biomass, well-known in ecology and described by the Verhulst equation (for $k_2 = 0$) and decrease in the biomass increment as the amount of the fragmentary material in the coastal zone that promotes biocenosis degradation grows.

The model described by Eqs. (1) and (2) can be considered as the asymptotic approximation following from hydrodynamic equations [17].

ANALYSIS OF THE EQUATIONS. PROBLEM SOLUTION

Passing to the dimensionless variables $t' = k_1 t$, $B' = B / B_{\max}$, and $W' = W / W_m$, we obtain the nonlinear dynamic system of second order

$$\begin{aligned} \frac{dW'}{dt'} &= -\sigma_1 W' + \sigma_2 B W' - \sigma_5 B' + \sigma_3, \\ \frac{dB'}{dt'} &= B'(1 - B') - \sigma_4 W', \end{aligned} \tag{3}$$

where

$$\sigma_1 = (1 / k_1)(aH\gamma + C_0 + C_{\min}) > 0,\ \sigma_2 = C_0 / k_1 > 0,$$

$$\sigma_3 = (aH\gamma / k_1) + (4 / k_1 W_m) > 0,\ \sigma_4 = k_2 W_m / (k_1 B_{\max}) > 0,$$

$$\sigma_5 = \delta B_{\max} / (W_m k_1 > 0).$$

The coordinates of singular points of system (3) can be found from the relations

$$\begin{aligned} W'_* &= B'_* (1 - B'_*) / \sigma_4, \\ B'_* &= (\sigma_1 W'_* - \sigma_3) / (\sigma_5 + \sigma_2 W'_*), \end{aligned} \tag{4}$$

where

$$0 \le B'_* \le (\sigma_1 - 4\sigma_3\sigma_4) / \sigma_2,\ 0 \le W'_* \le 1/(4\sigma_4) \le 1,$$

$$0 \le \sigma_3 / \sigma_1 \le W'_* \le \sigma_3 / (\sigma_1 - \sigma_2) \le 1,\ \sigma_1 > \sigma_2.$$

Applying the parameters of system (3), we find the values of W'_* from the solution of the cubic equation

$$(W'_*)^3 + (W'_*)^2 \frac{[2\sigma_2\sigma_4\sigma_5 + \sigma_1(\sigma_1 - \sigma_2)]}{\sigma_2^2\sigma_4} + W'_* \frac{[\sigma_4\sigma_5 - \sigma_1\sigma_5 + \sigma_3(\sigma_2 - 2\sigma_1)]}{\sigma_2^2\sigma_4} + \frac{\sigma_3^2 + \sigma_3\sigma_5}{\sigma_2^2\sigma_4} = 0. \tag{5}$$

The matrix of the linearized system (3) has the form

$$\widetilde{A} = \begin{bmatrix} -\sigma_1 + \sigma_2 B'_* & \sigma_2 W'_* + \sigma_5 \\ -\sigma_4 & 1 - 2B'_* \end{bmatrix}. \tag{6}$$

The condition $\det \widetilde{A} = 0$ can be used to find the bifurcation set for saddle-type points (boundary of the saddles) [18, 19]. To find the Hopf bifurcation, we equate the trace of matrix (6) to zero:

$$\operatorname{tr} \widetilde{A} = 1 - \sigma_1 - 2B'_* + \sigma_2 B'_* = 0,$$

whence

$$0 \le B'_* = (\sigma_1 - 1)/(\sigma_2 - 2),\ 0 \le \sigma_2 \le \sigma_1 \le 1, \tag{7}$$

$$\det \widetilde{A} = \frac{(\sigma_2 - \sigma_1)\sigma_1(\sigma_2 + 4) + \sigma_2(1 - 2\sigma_2) + \lambda_0\sigma_5(\sigma_2 - 2)^2}{(\sigma_2 - 2)^2}, \tag{8}$$

$$\lambda_0 = \gamma_{4bif} = \frac{(\sigma_1 - 1)(\sigma_2 - \sigma_1 - 1)(\sigma_2 - 2\sigma_1)}{(\sigma_2 - 2)[\sigma_3(\sigma_2 - 2) + \sigma_5(\sigma_1 - 1)]}. \tag{9}$$

Relation (9) (stability boundary of nodes and focal points [19]) is obtained from (4), (7).

Taking into account inequalities (7) and $\lambda_0 > 0$, we get

$$\sigma_3(\sigma_2 - 2) + \sigma_5(\sigma_1 - 1) < 0 \Leftrightarrow \sigma_3 > \frac{(1 - \sigma_1)\sigma_5}{(\sigma_2 - 2)} < 0,$$

whence the inequality $\sigma_2\sigma_3 + \sigma_1\sigma_5 > 0$ follows, which is useful for the further analysis.

For the Hopf bifurcation to exist, it is necessary that $\det \widetilde{A} > 0$ and $0 \le \sigma_2 < \sigma_1 \le 1$, which is equivalent to the inequality

$$\frac{-(4\sigma_1 - \sigma_1^2 + 1 - 4\lambda_0\sigma_5) + D^{1/2}}{2(\sigma_1 - 2 + \lambda_0\sigma_5)} < \sigma_2 < \sigma_1 \tag{10}$$

for $\sigma_2 - 2 + \lambda_0\sigma_5 < 0$ and $0 \le \sigma_2 < \sigma_1 \le 1$ for $\sigma_2 - 2 + \lambda_0\sigma_5 > 0$ (here also $\lambda_0\sigma_5 \ge 1$), where $D = (4\sigma_1 - \sigma_1^2 + 1 - 4\lambda_0\sigma_5)^2 + 16(\sigma_1 - \lambda_0\sigma_5)(\sigma_1 - 2 + \lambda_0\sigma_5)$.

As the parameter σ_5 increases beginning from zero (no inflow of biogenic material), curvilinear field ($\det \widetilde{A} > 0$) in the triangle $0 \le \sigma_2 < \sigma_1 \le 1$ gradually increases and covers it completely for $\sigma_2 - 2 + \lambda_0 \sigma_5$. The bifurcation happens on the boundary bifurcation curve (9), which passes through the specified field of parameters σ_1 and σ_2.

For the further mathematical analysis, we introduce the following notation: $B' = x$, $W' = y$, $t' = t$, $B'_* = x_*$, $W'_* = y_*$, $x' = x - x_*$, $y' = y - y_*$, and $\mu = \lambda - \lambda_0 = \sigma_1 - \sigma_{4bif}$. Then system (3) becomes

$$\frac{dx'}{dt} = x'(1 - 2x_*) - (\mu + \lambda_0) y' - (x')^2,$$
$$\frac{dy'}{dt} = (\sigma_2 y_* + \sigma_5) x' + (\sigma_2 x_* - \sigma_1) y' + \sigma_2 x' y'. \tag{11}$$

In the notation from [20] we can write

$$X_\mu(x', y') = (x'(1 - 2x_*) - (\mu + \lambda_0) y' - (x')^2,\ (\sigma_2 y_* + \sigma_5) x' + (\sigma_2 x_* - \sigma_1) y' + \sigma_2 x' y'),$$
$$dX_\mu(0,0) = \begin{bmatrix} 1 - 2x_* & -(\mu + \lambda_0) \\ \sigma_2 y_* + \sigma_5 & \sigma_2 x_* - \sigma_1 \end{bmatrix},\ dX_0(0,0) = \begin{bmatrix} A & -\lambda \\ k & -A \end{bmatrix},$$

where

$$1 - 2x_* = \frac{\sigma_2 - 2\sigma_1}{\sigma_2 - 2} = A > 0,\ \sigma_2 x_* - \sigma_1 = -A < 0,$$
$$\sigma_2 y_* + \sigma_5 = \frac{\sigma_2(\sigma_1 - 1)(\sigma_2 - \sigma_1 - 1)}{(\sigma_2 - 2)^2 \lambda_0} + \sigma_5 = k \text{ for } \mu = 0 \ (\operatorname{tr} \widetilde{A} = 0).$$

The determinant of matrix $dX_0(0,0)$ coincides with formula (8). The normalized Jacobi matrix becomes

$$dX_0(0,0) = \begin{bmatrix} 1 & -\lambda_0 / A \\ k / A & -1 \end{bmatrix}.$$

The eigenvalues of the Jacobi matrix have the form

$$\lambda_{1,2} = \frac{1}{2} - (4 \det \widetilde{A})^{1/2} = \pm i (\det \widetilde{A})^{1/2} = \pm i\gamma = \pm i\omega_0,$$

where $\det \widetilde{A}$ can be found from formulas (8) and (9).

The eigenvector of matrix $dX_0(0,0)$ for the eigenvalue $\lambda_1 = i\gamma = i\omega_0$ can be found from the matrix equation

$$(dX_0(0,0) - \lambda_1 I) \times U_1 = 0, \tag{12}$$

where I is unit matrix and U_1 is the eigenvector,

$$U_1 = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} + i \begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix}.$$

Equation (12) yields the eigenvector normalized so that its first nonzero component is one:

$$U_1 = \hat{e}_1 + i\hat{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} -A/\gamma \\ -k/\gamma \end{bmatrix}, \tag{13}$$

whence

$$X_0(x'\hat{e}_1 + y'\hat{e}_2) = X_0(x' - (A/\gamma) y'),\ \ y' = y'(-k/\gamma)$$

and hence,

$$X_0(x', y') = ((x' - (A/\gamma) y') A + (\lambda_0 k / \gamma) y' - (x' - (A/\gamma) y'))^2,$$

$$k((x' - (A/\gamma)y') + (k/\gamma)Ay' - \sigma_2(x' - (A/\gamma)y')(k/\gamma)y') \tag{14}$$
$$= ((x' - (A/\gamma)y')A + (\lambda_0 k/\gamma)y' - (x' - (A/\gamma)y'))^2,$$
$$kx' + (\sigma_2 kA/\gamma^2)(y')^2 = (\sigma_2 k/\gamma)x'y'.$$

The expansion of $X_0(x', y')$ in the new basis has the form

$$\widetilde{A}(x', y')\begin{bmatrix}1\\0\end{bmatrix} + \widetilde{B}(x', y')\begin{bmatrix}-A/\gamma\\-k/\gamma\end{bmatrix} = X_0(x', y'),$$

whence

$$\begin{aligned} \widetilde{A}(x', y') - \widetilde{B}(x', y')(A/\gamma) &= (x' - (A/\gamma)y'A + (\lambda_0 k/\gamma)y' - (x' - (A/\gamma)y'))^2, \\ -\widetilde{B}(x', y')(k/\gamma) &= kx' + (\gamma_2 kA/\gamma^2)(y')^2 - (\sigma_2 k/\gamma)x'y'. \end{aligned} \tag{15}$$

System (15) yields

$$\widetilde{A}(x', y') = (\lambda_0 kA^2/\gamma)y' + (A/\gamma)(2+\sigma_2)x'y' - (x')^2 - (A/\gamma)^2(1+\sigma_2)(y')^2. \tag{16}$$

In the new coordinate system the expression $X_0(x', y')$ becomes

$$X_0(x', y') = (\widetilde{A}(x', y'), \widetilde{B}(x', y')),$$

where $\widetilde{A}$ and $\widetilde{B}$ are defined by (15) and (16).

Let us show that $\lambda_0 k - A^2 = \gamma^2 = \det \widetilde{A}$. This follows from the expressions for k and from formula (8) for $\sigma_3 = \sigma_5$:

$$(\sigma_2 - \sigma_1)\sigma_1(\sigma_2 + 4) + \sigma_2(1 - 2\sigma_2) = \sigma_2(\sigma_1 - 1)(\sigma_2 - \sigma_1 - 1) - (\sigma_2 - 2\sigma_1)^2.$$

Thus, the initial expressions for calculating the stability criterion have the form

$$X' = \widetilde{A}(x', y') = \gamma y' + (A/\gamma)(2+\sigma_2)x'y' - (x')^2 - (A/\gamma)^2(1+\sigma_2)(y')^2, \tag{17}$$

$$X_1^2 = \widetilde{B}(x', y') = -\gamma x' + \sigma_2 x'y' - (\sigma_2 A/\gamma)(y')^2.$$

The Jacobi matrix in the new basis becomes diagonal

$$dX_0(0,0) = \begin{bmatrix}0 & \gamma\\ -\gamma & 0\end{bmatrix}. \tag{18}$$

The stability criterion according to [20] has the form

$$V'''(0) = \frac{3\pi}{4\gamma^2}\left[-\frac{\partial^2 X'}{\partial x'^2}\frac{\partial^2 X'}{\partial x'\partial y'} + \frac{\partial X^2}{\partial y'^2}\frac{\partial^2 X^2}{\partial x'\partial y'} - \frac{\partial^2 X'}{\partial y'^2}\frac{\partial^2 X'}{\partial x'\partial y'} - \frac{\partial^2 X'}{\partial y'^2}\frac{\partial^2 X^2}{\partial y'^2}\right], \tag{19}$$

where V is bias function related to the Poincare mapping.

Substituting expressions (17) into formula (19) yields

$$V'''(0) = \frac{3\pi A}{4\gamma^3}(1 + (a/\gamma)^2)(2 + \sigma_2 - \sigma_2^2). \tag{20}$$

Since the parameters A, γ, and $(2 + \sigma_2 - a\sigma_2^2)$ are positive, we obtain $V'''(0) > 0$; hence, the limit cycle is unstable [19].

The approximate analytical characteristics of the limit cycle can be found according to the methodology developed in [21]. In the notation of this study, the matrix

$$P = \begin{bmatrix} 1 & A/\gamma \\ 0 & k/\gamma \end{bmatrix}$$

is obtained, where $P = (\operatorname{Re} U_1 \; -\operatorname{Im} U_1)$ and U_1 is defined by formula (13), which introduces the linear transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & A/\gamma \\ 0 & k/\gamma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow x' = \tilde{x} + \frac{A}{\gamma} y', \;\; y' = \frac{k}{\gamma} \tilde{y},$$

to within the sign (see (13) and (14)), coinciding with the change of variables obtained from the use of the previous approach [20]. In the new variables, dynamic system (11) for $\mu = 0$ becomes

$$\begin{gathered} \frac{d\tilde{x}}{dt} = -\gamma \tilde{y} - (A/\gamma)(2+\sigma_2)\tilde{x}\tilde{y} - \tilde{x}^2 - (A/\gamma)^2(1+\sigma_2)\tilde{y}^2, \\ \frac{d\tilde{y}}{dt} = \gamma\tilde{x} + \sigma_2\tilde{x}\tilde{y} + (\sigma_2 A/\gamma)\tilde{y}^2. \end{gathered} \tag{21}$$

The eigenmatrix of the linearized system (21) (Jacobi matrix) has the form (compare with matrix (18))

$$\tilde{A}_1 = \begin{bmatrix} 0 & -\gamma \\ \gamma & 0 \end{bmatrix}.$$

Use system (20) to write the initial functions corresponding to formulas (17) of the approach [20], necessary to calculate the analytical characteristics of the limit cycle:

$$\begin{gathered} F_1(\tilde{x},\tilde{y}) = -\gamma \tilde{y} - (A/\gamma)(2+\sigma_2)\tilde{x}\tilde{y} - \tilde{x}^2 - (A/\gamma)^2(1+\sigma_2)\tilde{y}^2, \\ F_2(\tilde{x},\tilde{y}) = \gamma\tilde{x} + \sigma_2\tilde{x}\tilde{y} + (\sigma_2 A/\gamma)\tilde{y}^2 \cdot \end{gathered} \tag{22}$$

According to [21], write the expressions for functions:

$$g_{11} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial \tilde{x}^2} + \frac{\partial^2 F_1}{\partial \tilde{y}^2} + i\left(\frac{\partial^2 F_2}{\partial \tilde{x}^2} + \frac{\partial^2 F_2}{\partial \tilde{y}^2}\right)\right],$$

$$g_{02} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial \tilde{x}^2} - \frac{\partial^2 F_1}{\partial \tilde{y}^2} - 2\frac{\partial^2 F_2}{\partial \tilde{x}\partial \tilde{y}} + i\left(\frac{\partial^2 F_2}{\partial \tilde{x}^2} - \frac{\partial^2 F_2}{\partial \tilde{y}^2} + 2\frac{\partial^2 F_1}{\partial \tilde{x}\partial \tilde{y}}\right)\right],$$

$$g_{20} = \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial \tilde{x}^2} - \frac{\partial^2 F_1}{\partial \tilde{y}^2} + 2\frac{\partial^2 F_2}{\partial \tilde{x}\partial \tilde{y}} + i\left(\frac{\partial^2 F_2}{\partial \tilde{x}^2} - \frac{\partial^2 F_2}{\partial \tilde{y}^2} - 2\frac{\partial^2 F_1}{\partial \tilde{x}\partial \tilde{y}}\right)\right], \;\; g_{21} = 0.$$

Then taking into account (22) we obtain

$$\begin{gathered} g_{11} = -\frac{1}{2} - \frac{A^2(\sigma_2+1)}{2\gamma^2} + \frac{i\sigma_2 A}{2\gamma}, \\ g_{02} = \frac{(\sigma_2+1)}{2}\left(\frac{A^2}{\gamma^2} - 1\right) - i\frac{A}{\gamma}(\sigma_2+1), \\ g_{20} = \frac{1}{2}(\sigma_2-1) + \frac{A}{2\gamma^2}(\sigma_2+1) + i\frac{A}{\gamma}, \;\; g_{21} = 0. \end{gathered} \tag{23}$$

According to [21], we use (23) to calculate the expressions

$$C_1(0)=\frac{i}{2\gamma}\left(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2\right)+\frac{g_{21}}{2},$$

$$\operatorname{Re}C_1(0)=\frac{A}{8\gamma^2}\left(1+\frac{A^2}{\gamma^2}\right)(2+\sigma_2-\sigma_2^2)>0,$$

$$\operatorname{Im}C_1(0)=-\frac{(\sigma_2+1)}{8\gamma^3}(4A^2+\gamma^2+3A^2\sigma_2)-\frac{1}{24}(\sigma_2+1)^2\left(1+\frac{2A^2}{\gamma^2}+\frac{10A^4}{\gamma^4}\right)<0.$$

The main analytical characteristics of the limit cycle can be found from formulas

$$\mu_2=-\operatorname{Re}C_1(0)/\alpha'(0),\ \ \beta_2=2\operatorname{Re}C_1(0)>0,$$

$$\tau_2=-\frac{1}{\gamma}\left[\operatorname{Im}C_1(0)-\frac{\operatorname{Re}C_1(0)}{\alpha'(0)}\omega'(0)\right],$$

$$\alpha'(0)=\left.\frac{d\alpha}{d\lambda}\right|_{\lambda=\lambda_0}=\frac{1}{2}(\sigma_2-2)\frac{(\sigma_2\sigma_3+\sigma_1\sigma_5)}{k^2}\left.\frac{dy}{d\lambda}\right|_{\lambda=\lambda_0},$$

$$\left.\frac{dy_*}{d\lambda}\right|_{\lambda=\lambda_0}=\frac{(k-\sigma_5)^2\sigma_1(\sigma_1-\sigma_2)+\sigma_2(k-\sigma_5)[\sigma_3(\sigma_2-2\sigma_1)-\sigma_1\sigma_5]+\sigma_2^2\sigma_3(\sigma_3+\sigma_5)}{\sigma_2\lambda_0\{3(k-\sigma_5)^2\sigma_2\lambda_0+2(k-\sigma_5)[2\sigma_2\sigma_5\lambda_0+\sigma_1(\sigma_1-\sigma_2)]+\sigma_2[\lambda_0\sigma_5^2-\sigma_1\sigma_5+\sigma_3(\sigma_2-2\sigma_1)]\}},$$

$$\omega'(0)=\left.\frac{d\omega}{d\lambda}\right|_{\lambda=\lambda_0}=\frac{1}{2}\,\frac{[A(\sigma_2+2)(\sigma_2\sigma_3+\sigma_1\sigma_5)k+\lambda_0\sigma_2]\left.\frac{dy_*}{d\lambda}\right|_{\lambda=\lambda_0}+k}{\det\widetilde{A}(\det\widetilde{A})^{1/2}},$$

$$T=\frac{2\pi}{\gamma}(1+\tau_2\varepsilon^2+O(\varepsilon^4)),\ \ \varepsilon^2=\frac{\lambda-\lambda_0}{\mu_2}+O(\lambda-\lambda_0)^2,\ \ \alpha(\lambda)=\operatorname{Re}\lambda_1=\frac{1}{2}\operatorname{tr}\widetilde{A},$$

$$\omega(\lambda)=\operatorname{Im}\lambda_1=(1/2)[4\det\widetilde{A}-(\operatorname{tr}\widetilde{A})^2]^{1/2}>0,\ \ \lambda_1=\lambda_2=\alpha(\lambda)+i\omega(\lambda),$$

where the matrix $\widetilde{A}$ has the form (6).

Since the Floquet parameter is greater than zero, $\beta_2>0$, the limit cycle is unstable, which agrees with the use of the Marsden and McCraken stability criterion (19), (20). The limit cycle period T, as σ_5 increases from 0 to ∞ (for $\varepsilon^2\approx 0$), decreases from $T_0=2\pi/\gamma_0$ to

$$T(\sigma_5=\infty)=2\pi/\gamma_0^2\left[\frac{(\sigma_2-\sigma_1-1)(\sigma_2-2\sigma_1)}{(\sigma_2-2)^2}\right]^{-1/2},$$

where γ_0^2 can be found from formula (8) for $\sigma_5=0$ (is obtained from relations (8) and (9) as $\sigma_5\to\infty$). For $\sigma_5=0$ it is shown that $\mu_2<0$ and, hence, the periodic solution exists for $\lambda<\lambda_0$ ("subcritical" bifurcation). To write the periodic solution to within the initial phase, we can use the procedure proposed in [21].

Let us find some estimates for the oscillations period. For $\sigma_5=0$ (no inflow of biogenic material), $k_1=0.5-0.6$ year^{-1} (in case of absent fragmentary material ($W=0$) and small amount of the biomass B, its increment is 50–60% per year), when k_1 and k_2 belong to the small segment (10) in the triangle $0\le k_2<k_1\le 1$, we obtain $\omega=\gamma_0\approx 0.1\div 0.2$, whence the dimensional period is $T=2\pi/(\gamma_0 k_1)\cong$ 50–100 years. For $\sigma_5>0$ near the value $\sigma_3=\sigma_5(1-\sigma_1)/(\sigma_2-2)<0$ (σ_3 varies in wide range due to the parameter U) it is possible to make λ_0 rather large (of the order of one). For real values of

σ_5 of the order of one and small expression $(\sigma_2 - \sigma_1)\sigma_1(\sigma_2 + 4) + \sigma_2(1 - 2\sigma_2)$ in the triangular domain $0 \le \sigma_2 < \sigma_1 \le 1$ we obtain $\gamma_0 = (\det \widetilde{A})^{1/2}$ of the same order ($\gamma_0 = 1 \div 2$), whence $T = 2\pi / (\gamma_0 k_1) \cong$ 5–10 years, which is ten times less than for $\sigma_5 = 0$.

The proposed model is an attempt to describe complex interaction processes in coastal ecogeosystems. There are ample opportunities to improve it: from taking into account the nonlinearity in Eq. (1) (cliff abrasion velocity and material abrasion intensity as functions of the volume of the material, etc.) up to including new mechanisms and the corresponding equations into the model, for example, for the velocity of seabed abrasion on the shelf. Applying the mathematical apparatus of evaluations of the Hopf bifurcations for such problem can stimulate the use of the analytical-computing technique outlined in [20–22] in searching and analyzing the Hopf bifurcations in mathematical models of various ecogeosystems.

The model can be tested by means of decreasing times, which will give information about the ecological situation for previous years for a long period of time.

CONCLUSIONS

In the present paper, we have applied the theory of dynamic systems to analyze the ecological state of seacoast abraded by surface gravity waves. We have carried out the qualitative analysis of abrasion of a coast as an ecological system on the basis of a simplified model described by the system of ordinary nonlinear differential equations. The applied semiempirical model is based on averaged quantities and can be considered as an approximation of the original hydrodynamic model. Unlike the well-known approaches, this model takes into account essentially nonlinear effects and control (beach feed). The analysis has been carried out by stability theory methods. Because of impaired ecological conditions, ecological equilibrium can be violated and passage to a new state (stable or unstable) is possible. The characteristics of limit cycle have been obtained and its stability has been analyzed.

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