EQUATIONS OF NONISOTHERMAL FILTRATION IN FAST PROCESSES IN ELASTIC POROUS MEDIA

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UDC 517.958:531.72, 517.958:539.3(4)

The problem of the nonisothermal joint motion of an elastic porous body and the fluid filling the pores is considered for the case where the duration of the physical process is fractions of a second. A rigorous derivation of averaged equations (equations not containing fast oscillating coefficients) based on the Nguetseng two-scale convergence method is proposed. For various combinations of physical parameters of the problem, these equations include anisotropic nonisothermal Stokes equations for the velocity of the fluid component and the equations of nonisothermal acoustics for the displacements of the solid component or anisotropic nonisothermal Stokes equations for a single-velocity continuum.

Key words: *nonisothermal Stokes and Lamé equations, hydraulic fracture, two-scale convergence, averaging of periodic structures.*

INTRODUCTION

In the present work, we propose a model for fast nonisothermal processes in an elastic deformable medium perforated by a system of channels and pores (elastic porous media) filled with a liquid or gas. The solid component of such media is called the soil skeleton, and the domain occupied by the fluid is called the pore space.

In the dimensionless (unprimed) variables

$$\boldsymbol{x}' = L\boldsymbol{x}, \qquad t' = \tau t, \qquad \boldsymbol{w}' = L\boldsymbol{w}, \qquad \theta' = \vartheta_* \frac{L}{\tau v_*}\,\theta,$$

the differential equations of the model for small deviations of the dimensionless displacements $\boldsymbol{w}$ and small deviations of the dimensionless temperature θ in the domain $\Omega \in \mathbb{R}^3$ at $t > 0$ are written as

$$\alpha_\tau \bar{\rho}\,\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \operatorname{div} \boldsymbol{P} + \bar{\rho}\boldsymbol{F}; \tag{1}$$

$$\alpha_\tau \bar{c}_p\,\frac{\partial \theta}{\partial t} = \operatorname{div}(\bar{\alpha}_\varkappa \nabla\theta) - \bar{\alpha}_\theta \frac{\partial}{\partial t}(\operatorname{div} \boldsymbol{w}) + \Psi; \tag{2}$$

$$p_f + \bar{\chi}\alpha_p \operatorname{div} \boldsymbol{w} = 0, \tag{3}$$

where the stress tensor of the continuous medium

$$\boldsymbol{P} = \bar{\chi}\boldsymbol{P}^f + (1-\bar{\chi})\boldsymbol{P}^s$$

coincides with the elastic stress tensor

$$\boldsymbol{P}^s = \alpha_\lambda \boldsymbol{D}(x, \boldsymbol{w}) + (\alpha_\eta \operatorname{div} \boldsymbol{w} - \alpha_{\theta s}\theta)\boldsymbol{I}$$

in the solid skeleton ($\boldsymbol{I}$ is a spherical tensor) and with the viscous stress tensor

$$\boldsymbol{P}^f = \alpha_\mu \boldsymbol{D}\Big(x, \frac{\partial \boldsymbol{w}}{\partial t}\Big) + (-p_f - \alpha_{\theta f}\theta)\boldsymbol{I}$$

in the pore space,

$$\boldsymbol{D}(x, \boldsymbol{u}) = (1/2)(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{t}}),$$

$$\bar{\rho} = \bar{\chi}\rho_f + (1-\bar{\chi})\rho_s, \qquad \bar{c}_p = \bar{\chi}c_{pf} + (1-\bar{\chi})c_{ps},$$

$$\bar{\alpha}_\varkappa = \bar{\chi}\alpha_{\varkappa f} + (1-\bar{\chi})\alpha_{\varkappa s}, \qquad \bar{\alpha}_\theta = \bar{\chi}\alpha_{\theta f} + (1-\bar{\chi})\alpha_{\theta s},$$

ρ_f and ρ_s are the average densities of the fluid and the solid skeleton, and c_{pf} and c_{ps} are the heat capacity coefficient for the fluid and the solid skeleton. The characteristic function $\bar{\chi}(\boldsymbol{x})$ of the pore spaces $\Omega_f \subset \Omega$ is considered known.

A derivation of Eqs. (1)–(3) and a description of all dimensionless constants (all of them are strictly positive) is contained in [1].

The problem is closed by the homogeneous initial and boundary conditions:

$$\boldsymbol{w}\Big|_{t=0} = 0, \quad \frac{\partial \boldsymbol{w}}{\partial t}\Big|_{t=0} = 0, \quad \theta\Big|_{t=0} = 0, \qquad \boldsymbol{x} \in \Omega; \tag{4}$$

$$\boldsymbol{w} = 0, \quad \theta = 0, \qquad \boldsymbol{x} \in S = \partial\Omega, \quad t \geq 0. \tag{5}$$

The mathematical model described by Eqs. (1)–(3) contains the natural small parameter ε, which is the ratio of the average pore size l to the characteristic dimension L of the domain considered:

$$\varepsilon = l/L.$$

Therefore, it is justified to determine the limiting regimes in the exact model as the small parameter tends to zero. This approximation considerably simplifies the initial problem, retaining all its basic properties. However, even in the presence of the small parameter, the problem remains difficult to solve and requires additional simplifying assumptions. From a geometrical point of view, as such a simplification one can use the assumption of periodicity of the pore space.

Assumption 1. Let the domain Ω be a periodic repetition of an elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)\times(0,1)\times(0,1)$; $1/\varepsilon$ is an integer such that Ω always contains an integer number of elementary cells Y^ε. We assume that Y_s is the solid part of the cell Y, its fluid part Y_f is an open complement of Y_s in Y, and the boundary $\gamma = \partial Y_f \cap \partial Y_s$ between the fluid and solid components is a Lipschitzian surface.

The pore space Ω_f^ε is a periodic repetition of an elementary cell εY_f, the solid skeleton Ω_s^ε is a periodic repetition of an elementary cell εY_s, and the Lipschitzian boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition of the boundary $\varepsilon\gamma$ in Ω.

The solid skeleton Ω_s^ε and the pore space Ω_f^ε are connected sets, and the section of the domain Ω_f^ε by an arbitrary plane $\{x_i = \mathrm{const},\ 0 < x_i < 1,\ i = 1,2,3\}$ is an open (in plane topology) set. In view of these assumptions, we have

$$\bar{\chi}(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi(\boldsymbol{x}/\varepsilon), \qquad \bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + (1-\chi^\varepsilon(\boldsymbol{x}))\rho_s,$$

$$\bar{c}_p = c_p^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})c_{pf} + (1-\chi^\varepsilon(\boldsymbol{x}))c_{ps},$$

$$\bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + (1-\chi^\varepsilon(\boldsymbol{x}))\rho_s,$$

$$\bar{\alpha}_\varkappa = \alpha_\varkappa^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\alpha_{\varkappa f} + (1-\chi^\varepsilon(\boldsymbol{x}))\alpha_{\varkappa s},$$

$$\bar{\alpha}_\theta = \alpha_\theta^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\alpha_{\theta f} + (1-\chi^\varepsilon(\boldsymbol{x}))\alpha_{\theta s},$$

where $\chi(\boldsymbol{y})$ is a characteristic function of Y_f in Y which defines the pore space. In the model considered, the function $\chi(\boldsymbol{y})$ is considered specified.

Let the dimensionless parameters given below depend on the small parameter of the problem ε and have finite or infinite limits

$$\lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon)\equiv\lim_{\varepsilon\searrow 0}\frac{2\mu}{\tau Lg\rho_0}=\mu_0,\qquad \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon)\equiv\lim_{\varepsilon\searrow 0}\frac{2\lambda}{Lg\rho_0}=\lambda_0,$$

$$\lim_{\varepsilon\searrow 0}\alpha_\tau(\varepsilon)\equiv\lim_{\varepsilon\searrow 0}\frac{L}{g\tau^2}=\tau_0,\qquad \lim_{\varepsilon\searrow 0}\alpha_p(\varepsilon)=p_*,$$

$$\lim_{\varepsilon\searrow 0}\alpha_\eta(\varepsilon)=\eta_0,\qquad \lim_{\varepsilon\searrow 0}\alpha_{\varkappa f}(\varepsilon)=\varkappa_{0f},\quad \lim_{\varepsilon\searrow 0}\alpha_{\varkappa s}(\varepsilon)=\varkappa_{0s},$$

$$\lim_{\varepsilon\searrow 0}\alpha_{\theta f}(\varepsilon)=\beta_{0f},\qquad \lim_{\varepsilon\searrow 0}\alpha_{\theta s}(\varepsilon)=\beta_{0s},\quad \lim_{\varepsilon\searrow 0}\frac{\alpha_\lambda}{\varepsilon^2}=\lambda_1,$$

where μ is the viscosity of the liquid (gas), λ is the Lamé constant, τ is the characteristic time of the process, ρ_0 is the density of water, and g is the acceleration due to gravity.

If $\tau_0=\infty$, the renormalization of the displacements and temperature

$$\boldsymbol{w}\to\alpha_\tau\boldsymbol{w},\qquad \theta\to\alpha_\tau\theta$$

reduces the problem to a similar problem, in which

$$\alpha_\tau=1,\qquad \alpha_\mu=\frac{2\mu\tau}{L^2\rho_0},\qquad \alpha_\lambda=\frac{2\lambda\tau^2}{L^2\rho_0}.$$

We note that the case $\tau_0=\infty$ occurs, for example, in descriptions of fast processes, such as hydraulic fracture of an oil bed, in which the duration of the process is a few fractions of a second.

In the case of isothermal motion, the most complete results were obtained in [2, 3]. Nonisothermal motion was considered in [4] under the constraints $\tau_0<\infty$, $\mu_0<\infty$, and $\lambda_0^{-1}<\infty$. In the present paper, which is a continuation of studies [2–4], we consider the case not studied earlier $\lambda_0=0$, in particular, the version

$$\tau_0=1,\qquad 0<\mu_0<\infty,\qquad \lambda_0=0.$$

It is shown that the averaged equations of the accurate model (1)–(5) is the anisotropic system of nonisothermal Stokes equations for the fluid component, which is related to the equations of acoustics for the solid component ($\lambda_1<\infty$), or the anisotropic system of nonisothermal Stokes equations for the single-velocity continuum ($\lambda_1=\infty$).

Obviously, in the solution of real physical problems, the presence of any limiting transitions is not assumed and there are only concrete physical constants (density of the medium, fluid viscosity, elastic constants of the solid skeleton, etc.) and two variables: the characteristic dimension of the domain considered L and the characteristic time of the physical process τ. By changing these variables in the region of applicability of the mathematical model, it is possible to determine the behavior of the dimensionless complexes α_μ, α_τ, α_λ, $\ldots$ which will allow one to choose a particular regime in the accurate model (1)–(5).

All necessary auxiliary statements and notation are given in [2].

1. FORMULATION OF THE MAIN RESULTS

As a rule, Eqs. (1) and (2) are considered within the framework of distribution theory. These equations are supplemented by the boundary conditions

$$[\boldsymbol{w}]=0,\qquad [\boldsymbol{P}\cdot\boldsymbol{n}]=0,\qquad \boldsymbol{x}_0\in\Gamma^\varepsilon,\quad t\geq 0; \tag{1.1}$$

$$[\theta]=0,\qquad [\alpha_\varkappa^\varepsilon\nabla\theta\cdot\boldsymbol{n}]=0,\qquad \boldsymbol{x}_0\in\Gamma^\varepsilon,\quad t\geq 0 \tag{1.2}$$

on the boundary Γ^ε, where $\boldsymbol{n}$ is the unit normal vector to the boundary;

$$[\varphi](\boldsymbol{x}_0)=\varphi_{(s)}(\boldsymbol{x}_0)-\varphi_{(f)}(\boldsymbol{x}_0),$$

$$\varphi_{(s)}(\boldsymbol{x}_0)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_s^\varepsilon}}\varphi(\boldsymbol{x}),\qquad \varphi_{(f)}(\boldsymbol{x}_0)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_f^\varepsilon}}\varphi(\boldsymbol{x}).$$

Condition (1.1) follows from the definition of the class of desired solutions — solutions (temperature θ and displacements $\boldsymbol{w}$) having the minimal continuity properties. The first condition in (1.2) is a corollary of the law of conservation of momentum at strong (contact) fractures, the second condition is a corollary of the energy conservation law.

There are various forms of Eqs. (1) and (2) and boundary conditions (1.1) and (1.2) which are equivalent in terms of distribution theory. In the present paper, it is reasonable to write them in the form of integral identities.

DEFINITION 1. The functions $(\boldsymbol{w}^\varepsilon, \theta^\varepsilon, p_f^\varepsilon, p_s^\varepsilon)$ are called a generalized solution of problem (1)–(4) if they satisfy the regularity conditions

$$\nabla \boldsymbol{w}^\varepsilon,\ \nabla\theta^\varepsilon,\ p_f^\varepsilon,\ p_s^\varepsilon \in L^2(\Omega_T)$$

in the domain $\Omega_T = \Omega \times (0,T)$, boundary conditions (4), the equations

$$\frac{1}{\alpha_p}\, p_f^\varepsilon = -\chi^\varepsilon \operatorname{div} \boldsymbol{w}^\varepsilon + \frac{\chi^\varepsilon}{m}\,\beta^\varepsilon; \tag{1.3}$$

$$\frac{1}{\alpha_\eta}\, p_s^\varepsilon = -(1-\chi^\varepsilon) \operatorname{div} \boldsymbol{w}^\varepsilon - \frac{1-\chi^\varepsilon}{1-m}\,\beta^\varepsilon \tag{1.4}$$

almost everywhere in the domain Ω_T, the integral identity

$$\int\limits_{\Omega_T} \Big(\rho^\varepsilon \alpha_\tau \boldsymbol{w}^\varepsilon \cdot \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2} - \chi^\varepsilon \alpha_\mu \boldsymbol{D}(x, \boldsymbol{w}^\varepsilon) : \boldsymbol{D}\Big(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}\Big) - \rho^\varepsilon \boldsymbol{F}\cdot\boldsymbol{\varphi}$$

$$+ [(1-\chi^\varepsilon)\alpha_\lambda \boldsymbol{D}(x, \boldsymbol{w}^\varepsilon) - (p_f^\varepsilon + p_s^\varepsilon + \alpha_\theta^\varepsilon \theta^\varepsilon)\boldsymbol{I}] : \boldsymbol{D}(x, \boldsymbol{\varphi})\Big)\, dx\, dt = 0 \tag{1.5}$$

for all smooth vector functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{x}, t)$ such that

$$\boldsymbol{\varphi}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in S,\ t > 0, \qquad \boldsymbol{\varphi}(\boldsymbol{x}, T) = \frac{\partial \boldsymbol{\varphi}}{\partial t}(\boldsymbol{x}, T) = 0, \quad \boldsymbol{x} \in \Omega,$$

and the integral identity

$$\int\limits_{\Omega_T} \Big((c_p^\varepsilon \alpha_\tau \theta^\varepsilon + \alpha_\theta^\varepsilon \operatorname{div} \boldsymbol{w}^\varepsilon)\,\frac{\partial \xi}{\partial t} - \alpha_\varkappa^\varepsilon \nabla\theta^\varepsilon \cdot \nabla\xi + \Psi\xi\Big)\, dx\, dt = 0 \tag{1.6}$$

for all smooth functions, $\xi = \xi(\boldsymbol{x}, t)$ such that

$$\xi(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in S,\ t > 0, \qquad \xi(\boldsymbol{x}, T) = 0, \quad \boldsymbol{x} \in \Omega.$$

We introduce a new unknown function p_s^ε, which, by analogy with the function p_f^ε, will be called the pressure in the solid skeleton. Equation (1.4) will be called the continuity equation for the solid component. The normalizing term

$$\beta^\varepsilon = \int\limits_\Omega \chi^\varepsilon \operatorname{div} \boldsymbol{w}^\varepsilon\, dx \quad \text{at} \quad p_* + \eta_0 = \infty, \qquad \beta^\varepsilon = 0 \quad \text{at} \quad p_* + \eta_0 < \infty$$

is chosen so that the condition

$$\int\limits_\Omega p_f^\varepsilon\, dx = \int\limits_\Omega p_s^\varepsilon\, dx = 0 \tag{1.7}$$

is satisfied for $p_* + \eta_0 = \infty$. This increase in the number of unknown functions, first, allows an easy estimation of the pressure even if $p_* = \infty$ (incompressible fluid phase) or $\eta_0 = \infty$ (incompressible solid phase), and second, simplifies the form of the averaged equations.

In (1.5), the notation $\boldsymbol{A} : \boldsymbol{B}$ denotes the convolution of two tensors of the second rank in both indices:

$$\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}(\boldsymbol{B}^* \cdot \boldsymbol{A}) = \sum_{i,j=1}^{3} A_{ij} B_{ji}.$$

Below, we use the following assumption.

ASSUMPTION 2. Let: 1) $\Psi, \partial\Psi/\partial t, |\boldsymbol{F}|, |\partial \boldsymbol{F}/\partial t| \in L^2(\Omega_T)$; 2) the nondimensional parameters satisfy the constraints

$$p_*^{-1}, \eta_0^{-1}, |\ln \mu_0|, \beta_{0f}, \beta_{0s}, |\ln \varkappa_{0f}|, |\ln \varkappa_{0s}| < \infty, \qquad \tau_0 = 1, \qquad \lambda_0 = 0.$$

Everywhere below, the parameters of the model can take all values admitted by the conditions of the theorems. For example, if $p_*^{-1}=0$ (incompressible fluid) or $\eta_0^{-1}=0$ (incompressible solid skeleton), the terms containing these quantities vanish in all equations.

We also note that the cases $p_*=0$ and $\eta_0=0$ are not considered in the present paper since they are of no interest from both mathematical and physical points of view.

The main results of the present work are the following theorems.

Theorem 1. *For all $\varepsilon>0$ in an arbitrary time interval $[0,T]$, problem (1)–(5) has a unique generalized solution and*

$$\max_{0\le t\le T}\Big(\Big\|\frac{\partial^2\boldsymbol{w}^\varepsilon}{\partial t^2}(t)\Big\|_{2,\Omega}+\Big\|\chi^\varepsilon\nabla\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega}+\sqrt{\alpha_\lambda}\,\Big\|(1-\chi^\varepsilon)\nabla\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega}\Big)\le C_0; \tag{1.8}$$

$$\max_{0\le t\le T}\Big(\|\theta^\varepsilon(t)\|_{2,\Omega}+\|\nabla\theta^\varepsilon(t)\|_{2,\Omega}\Big)\le C_0; \tag{1.9}$$

$$\max_{0\le t\le T}\Big(\||p_f^\varepsilon|+|p_s^\varepsilon|\|_{2,\Omega}\Big)\le C_0, \tag{1.10}$$

where the constant C_0 does not depend on the small parameter ε.

Theorem 2. *The functions $\partial\boldsymbol{w}^\varepsilon/\partial t$ admit the continuation of $\boldsymbol{v}^\varepsilon$ from the domain $\Omega_f^\varepsilon\times(0,T)$ to the domain Ω_T, so that the sequence $\{\boldsymbol{v}^\varepsilon\}$ converges to the function $\boldsymbol{v}$ strongly in the space $L^2(\Omega_T)$ and weakly in the space $L^2((0,T);W_2^1(\Omega))$. Similarly, the sequence $\{\theta^\varepsilon\}$ converges to the function θ strongly in $L^2(\Omega_T)$ and weakly in $L^2((0,T);\overset{\circ}{W}{}_2^1(\Omega))$. At the same time, the sequences $\{\boldsymbol{w}^\varepsilon\}$, $\{(1-\chi^\varepsilon)\boldsymbol{w}^\varepsilon\}$, $\{p_f^\varepsilon\}$, and $\{p_s^\varepsilon\}$ converge to the functions $\boldsymbol{w}$, $\boldsymbol{w}^s$, p_f, and p_s, respectively, weakly in $L^2(\Omega_T)$.*

I. *If $\lambda_1=\infty$, then $\partial\boldsymbol{w}^s/\partial t=(1-m)\boldsymbol{v}=(1-m)\partial\boldsymbol{w}/\partial t$ and the weak and strong limits p_f, p_s, θ, and $\boldsymbol{v}$ satisfy the following initial-boundary-value problem in Ω_T:*

$$\hat\rho\,\frac{\partial\boldsymbol{v}}{\partial t}+\nabla(p_f+p_s+\hat\beta_0\theta)-\hat\rho\boldsymbol{F}$$

$$=\operatorname{div}\Big(\mu_0\boldsymbol{A}_0^f:\boldsymbol{D}(x,\boldsymbol{v})+\boldsymbol{B}_0^fp_s+\boldsymbol{B}_1^f\theta+\boldsymbol{B}_3^f\operatorname{div}\boldsymbol{v}+\int\limits_0^t\boldsymbol{B}_2^f(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau\Big); \tag{1.11}$$

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t}+\boldsymbol{C}_0^f:\boldsymbol{D}(x,\boldsymbol{v})+a_0^fp_s+a_1^f\theta$$

$$+(a_3^f+m)\operatorname{div}\boldsymbol{v}+a_4^f\langle\theta\rangle_\Omega+\int\limits_0^t a_2^f(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau=0,$$

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t}+\frac{1}{\eta_0}\frac{\partial p_s}{\partial t}+\operatorname{div}\boldsymbol{v}=0;$$

$$\hat c_p\,\frac{\partial\theta}{\partial t}-\frac{\beta_{0f}}{p_*}\frac{\partial p_f}{\partial t}-\frac{\beta_{0s}}{\eta_0}\frac{\partial p_s}{\partial t}+(\beta_{0f}-\beta_{0s})(a_3^f+a_4^f)\Big\langle\frac{\partial\theta}{\partial t}\Big\rangle_\Omega=\operatorname{div}(\boldsymbol{B}^\theta\cdot\nabla\theta)+\Psi. \tag{1.12}$$

Here $m=\int\limits_Y\chi\,dy$ is the porosity, $\hat\rho=m\rho_f+(1-m)\rho_s$, $\hat\beta_0=m\beta_{0f}+(1-m)\beta_{0s}$, $\hat c_p=mc_{pf}+(1-m)c_{ps}$; the symmetric strictly positive definite tensor of the fourth rank $\boldsymbol{A}_0^f$, the matrices $\boldsymbol{C}_0^f$, $\boldsymbol{B}_0^f$, $\boldsymbol{B}_1^f$, $\boldsymbol{B}_3^f$, and $\boldsymbol{B}_2^f(t)$, the symmetric strictly positive definite matrix $\boldsymbol{B}^\theta$, and the scalar quantities a_0^f, a_1^f, a_3^f, a_4^f, and $a_2^f(t)$ are defined below.

The differential equations (1.11) and (1.12) are closed by the homogeneous initial and boundary conditions

$$\begin{gathered}\boldsymbol{v}(\boldsymbol{x},0)=0,\qquad \theta(\boldsymbol{x},0)=p_f(\boldsymbol{x},0)=p_s(\boldsymbol{x},0)=0,\qquad \boldsymbol{x}\in\Omega,\\ \boldsymbol{v}(\boldsymbol{x},t)=0,\qquad \theta(\boldsymbol{x},t)=0,\qquad \boldsymbol{x}\in S,\quad t>0.\end{gathered} \tag{1.13}$$

II. *If* $\lambda_1 < \infty$*, then, in the domain* Ω_T*, the weak and strong limits* $\boldsymbol{w}^s$*,* p_f*,* p_s*,* θ*, and* $\boldsymbol{v}$ *satisfy the initial-boundary-value problem which includes the anisotropic nonisothermal system of Stokes equations*

$$\rho_f m\,\frac{\partial \boldsymbol{v}}{\partial t} + \rho_s\,\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} + \nabla(p_f + p_s + \hat{\beta}_0\theta) - \hat{\rho}\boldsymbol{F}$$

$$= \operatorname{div}\Big(\mu_0 \boldsymbol{A}^f_0 : \boldsymbol{D}(x, \boldsymbol{v}) + \boldsymbol{B}^f_0 p_s + \boldsymbol{B}^f_1\theta + \boldsymbol{B}^f_3 \operatorname{div}\boldsymbol{v} + \int\limits_0^t \boldsymbol{B}^f_2(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau\Big),$$

$$\frac{1}{p_*}\,\frac{\partial p_f}{\partial t} + \boldsymbol{C}^f_0 : \boldsymbol{D}(x, \boldsymbol{v}) + a^f_0 p_s + a^f_1\theta$$

$$+ (a^f_3 + m)\operatorname{div}\boldsymbol{v} + a^f_4\langle\theta\rangle_\Omega + \int\limits_0^t a^f_2(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau = 0,$$

$$\frac{1}{p_*}\,\frac{\partial p_f}{\partial t} + \frac{1}{\eta_0}\,\frac{\partial p_s}{\partial t} + \operatorname{div}\boldsymbol{v} = 0,$$

$$\hat{c}_p\,\frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*}\,\frac{\partial p_f}{\partial t} - \frac{\beta_{0s}}{\eta_0}\,\frac{\partial p_s}{\partial t} + (\beta_{0f} - \beta_{0s})(a^f_3 + a^f_4)\Big\langle\frac{\partial \theta}{\partial t}\Big\rangle_\Omega = \operatorname{div}(\boldsymbol{B}^\theta\cdot\nabla\theta) + \Psi$$

for the velocity, pressure, and temperature in the fluid component, which is linked to the continuity equation

$$\frac{1}{p_*}\,\frac{\partial p_f}{\partial t} + \frac{1}{\eta_0}\,\frac{\partial p_s}{\partial t} + \operatorname{div}\frac{\partial \boldsymbol{w}^s}{\partial t} + m\operatorname{div}\boldsymbol{v} = 0$$

by the relation

$$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1-m)\boldsymbol{v}(\boldsymbol{x},t) + \int\limits_0^t \boldsymbol{B}^s_1(t-\tau)\tilde{\boldsymbol{z}}(\boldsymbol{x},\tau)\,d\tau,$$

$$\tilde{\boldsymbol{z}}(\boldsymbol{x},t) = -\frac{1}{1-m}\nabla p_s(\boldsymbol{x},t) - \beta_{0s}\nabla\theta + \rho_s\boldsymbol{F}(\boldsymbol{x},t) - \rho_s\,\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x},t) \tag{1.14}$$

in the case $\lambda_1 > 0$ *or by the momentum conservation law*

$$\rho_s\,\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s\boldsymbol{B}^s_2\,\frac{\partial \boldsymbol{v}}{\partial t} + ((1-m)I - \boldsymbol{B}^s_2)\Big(-\frac{1}{1-m}\nabla p_s - \beta_{0s}\nabla\theta + \rho_s\boldsymbol{F}\Big) \tag{1.15}$$

in the case $\lambda_1 = 0$ *for the displacements of the solid component. The problem is closed by the boundary and initial conditions* (1.13) *for the averaged temperatures of the entire medium and the velocity* $\boldsymbol{v}$ *of the fluid component and the homogeneous initial conditions and the edge condition*

$$\boldsymbol{w}^s(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \qquad (\boldsymbol{x},t)\in S, \quad t>0 \tag{1.16}$$

for the displacements $\boldsymbol{w}^s$ *of the solid component. In Eqs.* (1.14)–(1.16)*,* $\boldsymbol{n}(\boldsymbol{x})$ *is the unit normal to the boundary* S *at the point* $\boldsymbol{x}\in S$*; the matrices* $\boldsymbol{B}^s_1(t)$ *are defined below.*

2. PROOF OF THEOREM 1

To derive estimates (1.8) and (1.9),we consider the integral identity

$$\frac{d}{dt}\Big[\alpha_\tau\int\limits_\Omega\Big(\rho^\varepsilon\Big(\frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}\Big)^2 + c^\varepsilon_p\Big(\frac{\partial \theta^\varepsilon}{\partial t}\Big)^2\Big)\,dx + \alpha_\lambda\int\limits_\Omega(1-\chi^\varepsilon)\boldsymbol{D}\Big(x,\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) : \boldsymbol{D}\Big(x,\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)\,dx$$

$$+ \alpha_p\int\limits_\Omega\chi^\varepsilon\Big(\operatorname{div}\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)^2\,dx + \alpha_\eta\int\limits_\Omega(1-\chi^\varepsilon)\Big(\operatorname{div}\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)^2 dx\Big] + \int\limits_\Omega\alpha^\varepsilon_\varkappa\Big|\nabla\frac{\partial \theta^\varepsilon}{\partial t}\Big|^2\,dx$$

$$
+\alpha_\mu \int_\Omega \chi^\varepsilon \boldsymbol{D}\Big(x, \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}\Big) : \boldsymbol{D}\Big(x, \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}\Big) dx = \int_\Omega \frac{\partial \boldsymbol{F}}{\partial t} \cdot \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}\, dx
$$

$$
+\frac{\partial \beta^\varepsilon}{\partial t}\Big(\frac{\alpha_p}{m} \int_\Omega \chi^\varepsilon \operatorname{div} \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}\, dx + \frac{\alpha_\eta}{1-m} \int_\Omega (1-\chi^\varepsilon) \operatorname{div} \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}\, dx\Big), \tag{2.1}
$$

which is obtained after differentiation of the equations for $\boldsymbol{w}^\varepsilon$ and θ^ε with respect to time, multiplication of the first equation by $\partial^2 \boldsymbol{w}^\varepsilon/\partial t^2$ and the second equation by $\partial \theta^\varepsilon/\partial t$, their integration by parts, and summation.

If $p_* + \eta_0 < \infty$ ($\beta^\varepsilon = 0$), identity (2.1) leads to the estimate

$$
\max_{0<t<T} \Big(\Big\|\frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}(t)\Big\|_{2,\Omega} + \sqrt{\alpha_\lambda}\, \Big\|\nabla \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_s^\varepsilon} + \sqrt{\alpha_\eta}\, \Big\|\operatorname{div} \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_s^\varepsilon}
$$

$$
+ \sqrt{\alpha_p}\, \Big\|\operatorname{div} \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega_f^\varepsilon} + \Big\|\frac{\partial \theta^\varepsilon}{\partial t}(t)\Big\|_{2,\Omega}\Big) + \Big\|\Big|\nabla \frac{\partial \theta^\varepsilon}{\partial t}\Big| + \Big|\chi^\varepsilon \nabla \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2}\Big|\Big\|_{2,\Omega_T} \leq C_0, \tag{2.2}
$$

where C_0 does not depend on ε. Estimates (1.8) and (1.9) follow from (2.2), and estimate (1.10) for the pressures p_f^ε and p_s^ε follows from the continuity equations (1.3) and (1.4) and estimates (2.2).

Let $p_* + \eta_0 = \infty$. Then, estimates (1.8) and (1.9) follow from identity (2.1) when using the inequalities

$$
\frac{1}{m}\Big(\int_\Omega \chi^\varepsilon \operatorname{div} \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\, dx\Big)^2 \leq \int_\Omega \chi^\varepsilon \Big(\operatorname{div} \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)^2 dx,
$$

$$
\frac{1}{1-m}\Big(\int_\Omega (1-\chi^\varepsilon) \operatorname{div} \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\, dx\Big)^2 \leq \int_\Omega (1-\chi^\varepsilon)\Big(\operatorname{div} \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)^2 dx.
$$

Estimate (1.10) for the sum of the pressures $p_f^\varepsilon + p_s^\varepsilon$ follows from the basic integral identity (1.5) and estimates (1.8) and (1.9) as an estimate of the corresponding functional in $\overset{\circ}{W}{}_2^1(\Omega)$. Indeed, identity (1.5) written as

$$
\int_\Omega (p_f^\varepsilon + p_s^\varepsilon) \operatorname{div} \boldsymbol{\psi}\, dx = \int_\Omega \Big[\rho^\varepsilon\Big(\alpha_\tau \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2} - \boldsymbol{F}\Big) \cdot \boldsymbol{\psi} + \Big(\chi^\varepsilon \alpha_\mu \boldsymbol{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) + (1-\chi^\varepsilon)\alpha_\lambda \boldsymbol{D}(x, \boldsymbol{w}^\varepsilon) - \alpha_\theta^\varepsilon \theta^\varepsilon \boldsymbol{I}\Big) : \boldsymbol{D}(x, \boldsymbol{\psi})\Big]\, dx,
$$

and estimates (1.8) and (1.9) lead to

$$
\Big|\int_\Omega (p_f^\varepsilon + p_s^\varepsilon) \operatorname{div} \boldsymbol{\psi}\, dx\Big| \leq C_0 \max_{0 \leq t \leq T} \|\boldsymbol{\psi}(t)\|_{W_2^1(\Omega)}. \tag{2.3}
$$

Choosing $\boldsymbol{\psi}$ so as to satisfy the condition $p_f^\varepsilon + p_s^\varepsilon \equiv q = \operatorname{div} \boldsymbol{\psi}$, we obtain the required estimate for the sum of the pressures $p_f^\varepsilon + p_s^\varepsilon$. This choice is possible (see [5]) if we set

$$
\boldsymbol{\psi} = \nabla \varphi + \boldsymbol{\psi}_0,
$$

where

$$
\Delta \varphi = q, \quad \boldsymbol{x} \in \Omega, \qquad \varphi = 0, \quad \boldsymbol{x} \in \partial\Omega; \tag{2.4}
$$

$$
\operatorname{div} \boldsymbol{\psi}_0 = 0, \quad \boldsymbol{x} \in \Omega, \qquad \boldsymbol{\psi}_0 = -\nabla \varphi, \quad \boldsymbol{x} \in \partial\Omega. \tag{2.5}
$$

Indeed, estimate (2.3) leads to

$$
\int_\Omega q^2\, dx \leq C_0 \max_{0 \leq t \leq T} \|\boldsymbol{\psi}(t)\|_{W_2^1(\Omega)}.
$$

Continuing the solution of problem (2.4) in an odd manner through the boundary of the domain Ω, we obtain

$$
\varphi \in \overset{\circ}{W}{}_2^2(\Omega), \qquad \max_{0 \leq t \leq T} \|\nabla \varphi(t)\|_{W_2^1(\Omega)} \leq \max_{0 \leq t \leq T} \|q(t)\|_\Omega.
$$

We seek the solution $\boldsymbol{\psi}_0$ of problem (2.5) as a solution of the Stokes equations

$$
\Delta \boldsymbol{\psi}_0 + \nabla p = 0, \qquad \operatorname{div} \boldsymbol{\psi}_0 = 0, \qquad \boldsymbol{x} \in \Omega
$$

that satisfies the inhomogeneous boundary-value condition

$$\boldsymbol{\psi}_0 = -\nabla\varphi, \qquad \boldsymbol{x} \in \partial\Omega.$$

The latter problem has a unique solution such that

$$\max_{0\le t\le T} \|\boldsymbol{\psi}_0(t)\|_{W_2^1(\Omega)} \le \max_{0\le t\le T} \|\nabla\varphi(t)\|_{W_2^1(\Omega)}$$

if and only if

$$\int\limits_{\Omega} \operatorname{div}(\nabla\varphi)\, dx \equiv \int\limits_{\Omega} \Delta\varphi\, dx = \int\limits_{\Omega} q\, dx = 0.$$

It is easy to see that this condition follows from conditions (1.7). Thus, taking into account all estimates, we obtain the required estimate but only for the sum $p_f^\varepsilon + p_s^\varepsilon$. Because the product of these functions is equal to zero, this is sufficient to estimate each term.

Estimates (1.8)–(1.10) guarantee the existence and uniqueness of the generalized solution of problem (1)–(4). To prove this, it is sufficient to employ the Galerkin method, using the space $\mathring{W}_2^1(\Omega)$ as the basis space, and any basis orthonormalized in the scalar product of the space $L^2(\Omega)$ as the basis.

3. PROOF OF THEOREM 2

3.1. Weak and Two-Scale Limits of Sequences of Displacements and Pressures. By virtue of theorem 1, the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, and $\{\boldsymbol{w}^\varepsilon\}$ are uniformly (in the parameter ε) bounded in $L^2(\Omega_T)$. Hence, there exists a subsequence of $\{\varepsilon > 0\}$ and functions p_f, p_s, and $\boldsymbol{w}$ such that

$$p_f^\varepsilon \to p_f, \qquad p_s^\varepsilon \to p_s, \qquad \boldsymbol{w}^\varepsilon \to \boldsymbol{w}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Similarly, because the sequence $\{\theta^\varepsilon\}$ in $L^2\Big((0,T); W_2^1(\Omega)\Big)$ is bounded, there exist a subsequence of $\{\varepsilon > 0\}$ and a function $\theta \in L^2((0,T); \mathring{W}_2^1(\Omega))$ such that $\theta^\varepsilon \to \theta$ weakly in $L^2((0,T); \mathring{W}_2^1(\Omega))$ as $\varepsilon \searrow 0$.

Redenoting, if necessary, the indices, we assume that the sequences converge by themselves. We also note that

$$(1-\chi^\varepsilon)\alpha_\lambda D(x, \boldsymbol{w}^\varepsilon) \to 0$$

strongly in $L^2(\Omega_T)$ and that the sequence $\{\operatorname{div} \boldsymbol{w}^\varepsilon\}$ converges weakly to the function $\operatorname{div} \boldsymbol{w}$ in $L^2(\Omega_T)$ as $\varepsilon \searrow 0$.

Moreover, by virtue of the continuation lemma (see [2,‘6,‘7]), there exist functions

$$\boldsymbol{v}^\varepsilon \in L^\infty((0,T); W_2^1(\Omega)),$$

such that $\boldsymbol{v}^\varepsilon = \partial\boldsymbol{w}^\varepsilon/\partial t$ in $\Omega_f \times (0,T)$, $v^\varepsilon = 0$ on the part S_f^ε of the boundary S and

$$\Big\|\frac{\partial \boldsymbol{v}^\varepsilon}{\partial t}\Big\|_{2,\Omega_T} + \Big\|\nabla\frac{\partial \boldsymbol{v}^\varepsilon}{\partial t}\Big\|_{2,\Omega_T} \le C_0,$$

$$\max_{0\le t\le T}\Big(\|\boldsymbol{v}^\varepsilon(t)\|_{2,\Omega} + \|\nabla\boldsymbol{v}^\varepsilon(t)\|_{2,\Omega}\Big) \le C_0$$

(the constant C_0 does not depend on the small parameter ε).

Lemma 1. *There exist a subsequence of* $\{\varepsilon > 0\}$ *and a function* $\boldsymbol{v} \in L^\infty((0,T); \mathring{W}_2^1(\Omega))$ *such that* $\boldsymbol{v}^\varepsilon(\,\cdot\,,t) \to \boldsymbol{v}(\,\cdot\,,t)$ *weakly in* $W_2^1(\Omega)$ *as* $\varepsilon \searrow 0$ *for all* $t \in [0,T]$.

The proof Lemma 1 is rather standard.

From the Nguetseng theorem (see [2, 8]), it follows that there exist functions $P_f(\boldsymbol{x},t,\boldsymbol{y})$, $P_s(\boldsymbol{x},t,\boldsymbol{y})$, $\Theta(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})$ which are one-periodic in the variable $\boldsymbol{y}$ and are such that the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\nabla\theta^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\nabla\boldsymbol{v}^\varepsilon\}$ two-scale converge to the functions $P_f(\boldsymbol{x},t,\boldsymbol{y})$, $P_s(\boldsymbol{x},t,\boldsymbol{y})$, $\nabla\theta + \nabla_y\Theta(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, and $\nabla\boldsymbol{v} + \nabla_y\boldsymbol{V}(\boldsymbol{x},t,\boldsymbol{y})$, respectively.

3.2. Microscopic and Macroscopic Equations I. The following lemmas are valid.

Lemma 2. *For all $\boldsymbol{x} \in \Omega$ and $\boldsymbol{y} \in Y$, the weak two-scale limits of the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, and $\{\boldsymbol{v}^\varepsilon\}$ satisfy the relations*

$$P_s = p_s \frac{1-\chi}{1-m}, \qquad P_f = \chi P_f; \tag{3.1}$$

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t} + m \operatorname{div} \boldsymbol{v} + \langle \operatorname{div}_y \boldsymbol{V} \rangle_{Y_f} = \frac{\partial \beta}{\partial t}; \tag{3.2}$$

$$\frac{1}{p_*}\frac{\partial P_f}{\partial t} + \chi(\operatorname{div} \boldsymbol{v} + \operatorname{div}_y \boldsymbol{V}) = \frac{\chi}{m}\frac{\partial \beta}{\partial t}; \tag{3.3}$$

$$\frac{1}{p_*} p_f + \frac{1}{\eta_0} p_s + \operatorname{div} \boldsymbol{w} = 0; \tag{3.4}$$

$$\boldsymbol{w}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in S; \tag{3.5}$$

$$\operatorname{div}_y \boldsymbol{W} = 0; \tag{3.6}$$

$$\frac{\partial \boldsymbol{W}}{\partial t} = \chi \boldsymbol{v} + (1-\chi)\frac{\partial \boldsymbol{W}}{\partial t}, \tag{3.7}$$

where $\partial\beta/\partial t = \langle\langle \operatorname{div}_y \boldsymbol{V} \rangle_{Y_f}\rangle_\Omega$ if $p_* + \eta_0 = \infty$ and $\beta = 0$ if $p_* + \eta_0 < \infty$; $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal vector to the surface S at the point $\boldsymbol{x} \in S$.

The proof of the lemma is similar to the proof of the corresponding lemma in [2].

Corollary. Let $p_* + \eta_0 = \infty$. Then, the functions p_f and p_s satisfy the equalities

$$\langle p_f \rangle_\Omega = \langle p_s \rangle_\Omega = 0.$$

Lemma 3. *For all $(\boldsymbol{x}, t) \in \Omega_T$ and $y \in Y$, the relation*

$$\operatorname{div}_y \Big[\mu_0 \chi\Big(\boldsymbol{D}(y, \boldsymbol{V}) + \boldsymbol{D}(x, \boldsymbol{v})\Big) - \Big(\chi P_f + \beta_0(\boldsymbol{y})\theta + \frac{1-\chi}{1-m} p_s\Big)\boldsymbol{I}\Big] = 0, \tag{3.8}$$

where $\beta_0(\boldsymbol{y}) = \beta_{0f}\chi(\boldsymbol{y}) + \beta_{0s}(1 - \chi(\boldsymbol{y}))$, is valid.

Proof. In the integral identity (1.5), substituting a test function of the form $\boldsymbol{\varphi}^\varepsilon = \varepsilon\boldsymbol{\varphi}(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\varphi}(\boldsymbol{x}, t, \boldsymbol{y})$ is an arbitrary function which is one-periodic in $\boldsymbol{y}$ and vanishes on the boundary S and passing to the limit as $\varepsilon \searrow 0$, we obtain the required microscopic equation (3.8) on the cell Y.

Lemma 4. *Let $\hat{\rho} = m\rho_f + (1-m)\rho_s$ and $\hat{\beta}_0 = m\beta_{0f} + (1-m)\beta_{0s}$. Then, the functions $\boldsymbol{w}^s = \langle \boldsymbol{W} \rangle_{Y_s}$, $\boldsymbol{v}$, p_f, p_s in the domain Ω_T satisfy the system of macroscopic equations*

$$\rho_f m \frac{\partial \boldsymbol{v}}{\partial t} + \rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} - \hat{\rho}\boldsymbol{F} = \operatorname{div}\big[\mu_0(m\boldsymbol{D}(x, \boldsymbol{v}) + \langle \boldsymbol{D}(y, \boldsymbol{V})\rangle_{Y_f}) - (p_f + p_s + \hat{\beta}_0\theta)\boldsymbol{I}\big] \tag{3.9}$$

and the homogeneous initial conditions

$$\boldsymbol{w}^s(\boldsymbol{x}, 0) = 0, \qquad \Big(\rho_f m \boldsymbol{v} + \rho_s \frac{\partial \boldsymbol{w}^s}{\partial t}\Big)(\boldsymbol{x}, 0) = 0, \qquad \boldsymbol{x} \in \Omega.$$

Proof. Equations (3.9) and the corresponding initial conditions are obtained by passing to the limit in identity (1.5) if, as test functions, one uses functions independent of the fast variable $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$.

3.3. Microscopic and Macroscopic Equations II. The following lemmas are valid.

Lemma 5. *If $\lambda_1 = \infty$, the weak limits of the sequences $\{\boldsymbol{v}^\varepsilon\}$ and $\{\partial \boldsymbol{w}^\varepsilon/\partial t\}$ coincide:*

$$\boldsymbol{v} = \frac{\partial \boldsymbol{w}}{\partial t} = \frac{1}{1-m}\frac{\partial \boldsymbol{w}^s}{\partial t}.$$

Proof. Let $\Psi(\boldsymbol{x}, t, \boldsymbol{y})$ be an arbitrary smooth scalar function which is periodic in the variable $\boldsymbol{y}$. The sequence $\{\sigma_{ij}^\varepsilon\}$, where

$$\sigma_{ij}^\varepsilon = \int\limits_\Omega \sqrt{\alpha_\lambda}\frac{\partial w_i^\varepsilon}{\partial x_j}(\boldsymbol{x}, t)\Psi(\boldsymbol{x}, t, \boldsymbol{x}/\varepsilon)\, dx, \qquad \boldsymbol{w}^\varepsilon = (w_1^\varepsilon, w_2^\varepsilon, w_3^\varepsilon),$$

is uniformly bounded in the parameter ε. Hence,

$$\int\limits_{\Omega} \varepsilon \frac{\partial w_i^\varepsilon}{\partial x_j}(\boldsymbol{x},t)\Psi(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)\,dx = \frac{\varepsilon}{\sqrt{\alpha_\lambda}}\,\sigma_{ij}^\varepsilon \to 0$$

as $\varepsilon \searrow 0$, which is equivalent to the equality

$$\int\limits_{\Omega}\int\limits_{Y} W_i(\boldsymbol{x},t,\boldsymbol{y})\frac{\partial \Psi}{\partial y_j}(\boldsymbol{x},t,\boldsymbol{y})\,dx\,dy = 0, \qquad \boldsymbol{W} = (W_1,W_2,W_3)$$

or $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y}) = \boldsymbol{w}(\boldsymbol{x},t)$. By virtue of the last relation and the equality

$$\chi^\varepsilon\Big(\boldsymbol{v}^\varepsilon - \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) = 0$$

the limit $\partial \boldsymbol{w}/\partial t$ of the sequences $\{\partial \boldsymbol{w}^\varepsilon/\partial t\}$ coincides with the limit $\boldsymbol{v}$ of the sequence $\{\boldsymbol{v}^\varepsilon\}$.

Lemma 6. *Let $\lambda_1 < \infty$. Then, the weak two-scale limits p_s and $\boldsymbol{W}$ in the domain Y_s satisfy the microscopic equations*

$$\rho_s \frac{\partial^2 \boldsymbol{W}}{\partial t^2} = \lambda_1 \Delta_y \boldsymbol{W} - \nabla_y R + \boldsymbol{z}, \qquad \boldsymbol{y} \in Y_s; \tag{3.10}$$

$$\frac{\partial \boldsymbol{W}}{\partial t} = \boldsymbol{v}, \qquad \boldsymbol{y} \in \gamma \tag{3.11}$$

in the case of $\lambda_1 > 0$ and the microscopic equations

$$\rho_s \frac{\partial^2 \boldsymbol{W}}{\partial t^2} = -\nabla_y R + \boldsymbol{z}, \qquad \boldsymbol{y} \in Y_s; \tag{3.12}$$

$$\Big(\frac{\partial \boldsymbol{W}}{\partial t} - \boldsymbol{v}\Big)\cdot \boldsymbol{n} = 0, \qquad \boldsymbol{y} \in \gamma \tag{3.13}$$

in the case $\lambda_1 = 0$.

In (3.10), (3.12), *and* (3.13),

$$\boldsymbol{z} = -\frac{1}{1-m}\nabla p_s - \beta_{0s}\nabla\theta + \rho_s \boldsymbol{F}$$

and $\boldsymbol{n}$ is the unit normal to the boundary γ.

Equations (3.10) *and* (3.12) *are supplemented by the homogeneous initial conditions*

$$\boldsymbol{W}(\boldsymbol{y},0) = \frac{\partial \boldsymbol{W}}{\partial t}(\boldsymbol{y},0) = 0, \qquad \boldsymbol{y} \in Y_s.$$

Proof. As $\varepsilon \searrow 0$, the differential equations (3.10) and (3.12) and the corresponding initial conditions follow from the integral identity (1.5) with testing functions of the form $\boldsymbol{\psi} = \boldsymbol{\varphi}(x\varepsilon^{-1})h(\boldsymbol{x},t)$, where $\boldsymbol{\varphi}$ is a solenoidal finite function Y_s in the domain.

The boundary-value condition (3.11) is a consequence of the two-scale convergence of the sequence $\{\sqrt{\alpha_\lambda}\,\nabla \boldsymbol{w}^\varepsilon\}$ to the function $\sqrt{\lambda_1}\,\nabla_y \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$. By virtue of this convergence, the function $\nabla_y \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ is integrable in $L^2(Y)$. The boundary-value condition (3.13) follows from Eqs. (3.4) and (3.5).

Lemma 7. *For all $(\boldsymbol{x},t) \in \Omega_T$ and $y \in Y$, the strong two-scale limits θ and Θ satisfy the microscopic equation*

$$\mathrm{div}_y\,[\tilde{\varkappa}_0(\boldsymbol{y})(\nabla\theta + \nabla_y\Theta)] = 0, \tag{3.14}$$

where $\tilde{\varkappa}_0(\boldsymbol{y}) = \chi(\boldsymbol{y})\varkappa_{0f} + (1-\chi(\boldsymbol{y}))\varkappa_{0s}$.

The proof of the lemma is similar to the proof of Lemma 3.

Lemma 8. *For all $(\boldsymbol{x},t) \in \Omega_T$, the weak and strong limits of θ, p_f, and p_s satisfy the macroscopic heat-conduction equation*

$$\hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*}\frac{\partial p_f}{\partial t} - \frac{\beta_{0s}}{\eta_0}\frac{\partial p_s}{\partial t} + (\beta_{0f} - \beta_{0s})\frac{\partial \beta}{\partial t} = \mathrm{div}\,(\hat{\varkappa}_0\nabla\theta + \langle\tilde{\varkappa}_0\nabla_y\Theta\rangle_Y) + \Psi, \tag{3.15}$$

where $\hat{\varkappa}_0 = \langle\tilde{\varkappa}_0\rangle_Y$ and $\hat{c}_p = mc_{pf} + (1-m)c_{ps}$.

The proof of Lemma 8 is similar to the proof of Lemma 4 if, previously, the term $\alpha_\theta^\varepsilon \operatorname{div} \boldsymbol{w}^\varepsilon$ in identity (1.6) is expressed in terms of the pressure, using the continuity equations (1.3) and (1.4).

3.4. Averaged Equations I. We derive averaged equations for the fluid component.

Lemma 9. *If $\lambda_1 = \infty$, then $\partial \boldsymbol{w}/\partial t = \boldsymbol{v}$ and the strong and weak limits $\boldsymbol{v}$, p_f, and p_s in the domain Ω_T satisfy the system of averaged differential equations*

$$\hat{\rho}\,\frac{\partial \boldsymbol{v}}{\partial t} + \nabla(p_f + p_s + \hat{\beta}_0\theta) - \hat{\rho}\boldsymbol{F}$$

$$= \operatorname{div}\Big(\mu_0 \boldsymbol{A}_0^f : \boldsymbol{D}(x,\boldsymbol{v}) + \boldsymbol{B}_0^f p_s + \boldsymbol{B}_1^f\theta + \boldsymbol{B}_3^f \operatorname{div}\boldsymbol{v} + \int\limits_0^t \boldsymbol{B}_2^f(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau\Big); \tag{3.16}$$

$$\frac{1}{p_*}\,\frac{\partial p_f}{\partial t} + \boldsymbol{C}_0^f : \boldsymbol{D}(x,\boldsymbol{v}) + a_0^f p_s + a_1^f\theta$$

$$+ (a_3^f + m)\operatorname{div}\boldsymbol{v} + a_4^f\langle\theta\rangle_\Omega + \int\limits_0^t a_2^f(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau = 0; \tag{3.17}$$

$$\frac{1}{p_*}\,\frac{\partial p_f}{\partial t} + \frac{1}{\eta_0}\,\frac{\partial p_s}{\partial t} + \operatorname{div}\boldsymbol{v} = 0, \tag{3.18}$$

where the symmetric and strictly positive definite tensor of the fourth rank $\boldsymbol{A}_0^f$, the matrices $\boldsymbol{C}_0^f$, $\boldsymbol{B}_0^f$, $\boldsymbol{B}_1^f$, $\boldsymbol{B}_3^f$, and $\boldsymbol{B}_2^f(t)$, and the scalar quantities a_0^f, a_1^f, a_3^f, a_4^f, and $a_2^f(t)$ are defined below.

The differential equations (3.16) are supplemented by the homogeneous initial and boundary conditions

$$\boldsymbol{v}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x}\in\Omega, \qquad \boldsymbol{v}(\boldsymbol{x},t) = 0, \quad \boldsymbol{x}\in S, \quad t>0. \tag{3.19}$$

Proof. First of all, we note that, by virtue of Lemma 5, $\boldsymbol{v} = \partial\boldsymbol{w}/\partial t$.

The averaged equations (3.16) are obtained by substitution of the expressions

$$\mu_0\langle\boldsymbol{D}(y,\boldsymbol{V})\rangle_{Y_f} = \mu_0\boldsymbol{A}_1^f : \boldsymbol{D}(x,\boldsymbol{v}) + \boldsymbol{B}_0^f p_s + \boldsymbol{B}_1^f\theta$$

$$+ \boldsymbol{B}_3^f\operatorname{div}\boldsymbol{v} + \int\limits_0^t \boldsymbol{B}_2^f(t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau + \boldsymbol{A}(t)$$

into the macroscopic equations (3.9). In turn, the last formula is the result of solution of Eqs. (3.6) and (3.8) on the elementary cell Y_f. Indeed, if $p_* + \eta_0 < \infty$, then $\beta = 0$. Then, assuming that

$$\boldsymbol{V} = \sum_{i,j=1}^{3}\boldsymbol{V}^{(ij)}(\boldsymbol{y})D_{ij} + \boldsymbol{V}^{(0)}(\boldsymbol{y})p_s + \boldsymbol{V}^{(1)}(\boldsymbol{y})\theta + \int\limits_0^t \boldsymbol{V}^{(2)}(\boldsymbol{y},t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau,$$

$$P_f = \sum_{i,j=1}^{3}P^{ij}(\boldsymbol{y})D_{ij} + P^0(\boldsymbol{y})p_s + P^1(\boldsymbol{y})\theta + \int\limits_0^t P^{(2)}(\boldsymbol{y},t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)\,d\tau,$$

where

$$D_{ij}(\boldsymbol{x},t) = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j}(\boldsymbol{x},t) + \frac{\partial v_j}{\partial x_i}(\boldsymbol{x},t)\Big),$$

we obtain the following periodic boundary-value problems in the domain Y:

$$\operatorname{div}_y\,[\chi\boldsymbol{D}(y,\boldsymbol{V}^{(ij)}) - \chi P^{(ij)}\boldsymbol{I} + \chi J^{ij}] = 0, \qquad \chi\operatorname{div}_y\boldsymbol{V}^{(ij)} = 0,$$

$$\operatorname{div}_y\Big[\mu_0\chi\boldsymbol{D}(y,\boldsymbol{V}^{(0)}) - \Big(\chi P^{(0)} + \frac{1-\chi}{1-m}\Big)\boldsymbol{I}\Big] = 0, \qquad \chi\operatorname{div}_y\boldsymbol{V}^{(0)} = 0, \tag{3.20}$$

$$\operatorname{div}_y\,[\mu_0\chi\boldsymbol{D}(y,\boldsymbol{V}^{(1)})-(\beta_0(\boldsymbol{y})+\chi P^{(1)})\boldsymbol{I}]=0,\qquad \chi\operatorname{div}_y\boldsymbol{V}^{(1)}=0;$$

$$\operatorname{div}_y\,[\mu_0\chi\boldsymbol{D}(y,\boldsymbol{V}^{(2)})-\chi P^{(2)}\boldsymbol{I}]=0;$$

$$\frac{1}{p_*}\,\frac{\partial P^{(2)}}{\partial t}+\chi\operatorname{div}_y\boldsymbol{V}^{(2)}=0,\qquad \frac{1}{p_*}\,P^{(2)}(\boldsymbol{y},0)=-\chi(\boldsymbol{y}). \tag{3.21}$$

If $p_*=\infty$, then $\beta\neq 0$. Then, assuming that

$$\boldsymbol{V}=\sum_{i,j=1}^{3}\boldsymbol{V}^{(ij)}(\boldsymbol{y})D_{ij}+\boldsymbol{V}^{(0)}(\boldsymbol{y})p_s+\boldsymbol{V}^{(1)}(\boldsymbol{y})(\theta-\langle\theta\rangle_\Omega)+\boldsymbol{V}^{(3)}(\boldsymbol{y})\operatorname{div}\boldsymbol{v}+\boldsymbol{V}^{(4)}(\boldsymbol{y})\langle\theta\rangle_\Omega,$$

$$P_f=\sum_{i,j=1}^{3}P^{ij}(\boldsymbol{y})D_{ij}+P^{0}(\boldsymbol{y})p_s+P^{1}(\boldsymbol{y})(\boldsymbol{y})(\theta-\langle\theta\rangle_\Omega)+P^{(3)}(\boldsymbol{y})\operatorname{div}\boldsymbol{v}+P^{(4)}(\boldsymbol{y})\langle\theta\rangle_\Omega,$$

we obtain the following boundary-value problems for determining the functions $\{\boldsymbol{V}^{(3)},\,P^{(3)}\}$ and $\{\boldsymbol{V}^{(4)},\,P^{(4)}\}$:

$$\begin{gathered}\operatorname{div}_y\,[\mu_0\chi\boldsymbol{D}(y,\boldsymbol{V}^{(3)})-\chi P^{(3)}\boldsymbol{I}]=0,\quad \chi(\operatorname{div}_y\boldsymbol{V}^{(3)}+1)=0,\\ \operatorname{div}_y\,[\mu_0\chi\boldsymbol{D}(y,\boldsymbol{V}^{(4)})-(\chi P^{(4)}+\beta_0(\boldsymbol{y}))\boldsymbol{I}]=0,\quad \chi\operatorname{div}_y\boldsymbol{V}^{(4)}=(\chi/m)\langle\chi\operatorname{div}_y\boldsymbol{V}^{(4)}\rangle_Y.\end{gathered}\tag{3.22}$$

Finally, if $p_*<\infty$ and $\eta_0=\infty$, then

$$\boldsymbol{V}=\sum_{i,j=1}^{3}\boldsymbol{V}^{(ij)}(\boldsymbol{y})D_{ij}+\boldsymbol{V}^{(0)}(\boldsymbol{y})p_s+\boldsymbol{V}^{(1)}(\boldsymbol{y})(\theta-\langle\theta\rangle_\Omega)$$

$$+\int_0^t\boldsymbol{V}^{(2)}(\boldsymbol{y},t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)d\tau+\boldsymbol{V}^{(4)}(\boldsymbol{y})\langle\theta\rangle_\Omega,$$

$$P_f=\sum_{i,j=1}^{3}P^{ij}(\boldsymbol{y})D_{ij}+P^{0}(\boldsymbol{y})p_s+P^{1}(\boldsymbol{y})(\boldsymbol{y})(\theta-\langle\theta\rangle_\Omega)$$

$$+\int_0^t P^{(2)}(\boldsymbol{y},t-\tau)\operatorname{div}\boldsymbol{v}(\boldsymbol{x},\tau)d\tau+P^{(4)}(\boldsymbol{y})\langle\theta\rangle_\Omega.$$

Assumptions on the geometry of the elementary fluid cell Y_f guarantee the existence of a unique (to within a constant vector) solution of problems (3.20)–(3.22). To eliminate arbitrariness, we require that the following equalities be satisfied:

$$\langle\boldsymbol{V}^{(ij)}\rangle_{Y_f}=\langle\boldsymbol{V}^{(k)}\rangle_{Y_f}=\langle P^{(4)}\rangle_{Y_f}=0,\qquad i,j=1,2,3,\quad k=0,1,2,3,4.$$

Thus,

$$\boldsymbol{A}^f_0=m\boldsymbol{J}+\boldsymbol{A}^f_1,\qquad \boldsymbol{A}^f_1=\sum_{i,j=1}^{3}\langle\boldsymbol{D}(y,\boldsymbol{V}^{(ij)})\rangle_{Y_f}\otimes\boldsymbol{J}^{ij},$$

$$\boldsymbol{B}^f_i=\tilde{\boldsymbol{B}}^f_i,\quad i=0,1,2,\qquad \boldsymbol{B}^f_3=0\quad\text{at}\quad p_*<\infty,$$

$$\boldsymbol{B}^f_i=\tilde{\boldsymbol{B}}^f_i,\quad i=0,1,3,\qquad \boldsymbol{B}^f_2=0\quad\text{at}\quad p_*=\infty,$$

$$\tilde{\boldsymbol{B}}^f_i=\mu_0\langle\boldsymbol{D}(y,\boldsymbol{V}^{(i)})\rangle_{Y_f},\qquad i=0,1,2,3.$$

The symmetry of the tensor $\boldsymbol{A}^f_0$ is proved in [2].

Equations (3.17) and (3.18) for the pressures follow from Eqs. (3.2) and (3.4) and the equality

$$\langle \operatorname{div}_y \boldsymbol{V} \rangle_{Y_f} = \boldsymbol{C}_0^f : \boldsymbol{D}(x, \boldsymbol{v}) + a_0^f p_s + a_1^f \theta + a_3^f \operatorname{div} \boldsymbol{v}$$

$$+ a_4^f \langle \theta \rangle_\Omega + \int_0^t a_2^f (t - \tau) \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, \tau)\, d\tau,$$

where

$$\boldsymbol{C}_0^f = \sum_{i,j=1}^{3} \langle \operatorname{div}_y \boldsymbol{V}^{(ij)} \rangle_{Y_f} \boldsymbol{J}^{ij},$$

$$a_i^f = \tilde{a}_i^f, \quad i = 0, 1, 2, \qquad a_j^f = 0, \quad j = 3, 4 \quad \text{at} \quad p_* + \eta_0 < \infty,$$

$$a_i^f = \tilde{a}_i^f, \quad i = 0, 1, 3, 4, \qquad a_2^f = 0 \quad \text{at} \quad p_* = \infty,$$

$$a_i^f = \tilde{a}_i^f, \quad i = 0, 1, 2, 4, \qquad a_3^f = 0 \quad \text{at} \quad p_* < \infty, \quad \eta_0 = \infty,$$

$$\tilde{a}_i^f = \langle \operatorname{div}_y \boldsymbol{V}^{(i)} \rangle_{Y_f}, \quad i = 0, 1, 2, 3, \qquad \tilde{a}_4^f = \langle \operatorname{div}_y \boldsymbol{V}^{(4)} - \operatorname{div}_y \boldsymbol{V}^{(1)} \rangle_{Y_f}.$$

3.5. Averaged Equations II. The proof of Theorem 2 is completed by the derivation of averaged equations for displacements of the solid component.

Let $\lambda_1 < \infty$. As above, the limit $\boldsymbol{v}$ of the sequence $\{\boldsymbol{v}^\varepsilon\}$ satisfies an initial-boundary-value problem similar to (3.16)–(3.19). The main difference is that the weak limit $\partial \boldsymbol{w}/\partial t$ of the sequence $\{\partial \boldsymbol{w}^\varepsilon/\partial t\}$, generally speaking, is different from $\boldsymbol{v}$ since the following lemma is valid.

Lemma 10. *Let $\lambda_1 < \infty$. Then, the strong and weak limits $\boldsymbol{v}$, $\boldsymbol{w}^s$, p_f, and p_s of the sequences $\{\boldsymbol{v}^\varepsilon\}$, $\{(1 - \chi^\varepsilon)\boldsymbol{w}^\varepsilon\}$, $\{p_f^\varepsilon\}$, and $\{p_s^\varepsilon\}$ in the domain Ω_T satisfy the system of the differential equations consisting of the momentum conservation law*

$$\rho_f m \frac{\partial \boldsymbol{v}}{\partial t} + \rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} + \nabla(p_f + p_s + \hat{\beta}_0 \theta) - \hat{\rho} \boldsymbol{F}$$

$$= \operatorname{div} \Big(\mu_0 \boldsymbol{A}_0^f : \boldsymbol{D}(x, \boldsymbol{v}) + \boldsymbol{B}_0^f p_s + \boldsymbol{B}_1^f \theta + \boldsymbol{B}_3^f \operatorname{div} \boldsymbol{v} + \int_0^t \boldsymbol{B}_2^f (t - \tau) \operatorname{div} \boldsymbol{v}(\boldsymbol{x}, \tau)\, d\tau \Big), \tag{3.23}$$

the continuity equation (3.17) *for the velocity and pressure in the fluid component ($\boldsymbol{A}_0^f$, $\boldsymbol{B}_0^f$, $\boldsymbol{B}_1^f$, $\boldsymbol{B}_2^f$, $\boldsymbol{B}_3^f$ are defined in Lemma* 9*), the continuity equation*

$$\frac{1}{p_*} \frac{\partial p_f}{\partial t} + \frac{1}{\eta_0} \frac{\partial p_s}{\partial t} + \operatorname{div} \frac{\partial \boldsymbol{w}^s}{\partial t} + m \operatorname{div} \boldsymbol{v} = 0, \tag{3.24}$$

the relation

$$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1 - m) \boldsymbol{v}(\boldsymbol{x}, t) + \int_0^t \boldsymbol{B}_1^s (t - \tau) \cdot \tilde{\boldsymbol{z}}(\boldsymbol{x}, \tau)\, d\tau,$$

$$\tilde{\boldsymbol{z}}(\boldsymbol{x}, t) = \boldsymbol{z}(\boldsymbol{x}, t) - \rho_s \frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x}, t) \tag{3.25}$$

in the case $\lambda_1 > 0$ or the momentum conservation law in the form

$$\rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s \boldsymbol{B}_2^s \frac{\partial \boldsymbol{v}}{\partial t} + ((1 - m) I - \boldsymbol{B}_2^s) \boldsymbol{z} \tag{3.26}$$

in the case $\lambda_1 = 0$ for the displacements of the solid component. The problem is supplemented by the initial and boundary conditions (3.19) *for the velocity $\boldsymbol{v}$ of the fluid component and the homogeneous initial conditions and the boundary condition*

$$\boldsymbol{w}^s(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \qquad (\boldsymbol{x}, t) \in S, \quad t > 0 \tag{3.27}$$

for the displacements $\boldsymbol{w}^s$ of the solid component. In Eqs. (3.23)–(3.27), $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal to the boundary S at the point $\boldsymbol{x} \in S$; the matrices $\boldsymbol{B}_1^s(t)$ and $\boldsymbol{B}_2^s$ are defined below.

Proof. The boundary condition (3.27) follows from Eq. (3.5), the equality

$$\frac{\partial \boldsymbol{w}}{\partial t} = \frac{\partial \boldsymbol{w}^s}{\partial t} + m\boldsymbol{v},$$

and the homogeneous boundary conditions for the velocity of the fluid component $\boldsymbol{v}$.

This equality and Eq. (3.4) prove Eq. (3.24). Equations (3.23) are derived similarly. We derive averaged equations of motion for the displacements $\boldsymbol{w}^s$ of the solid component.

1. Let $\lambda_1 > 0$. Then, the solution of the microscopic equations (3.6), (3.10), and (3.11) supplemented by homogeneous initial conditions is found from the formulas

$$\boldsymbol{W} = \int_0^t \Big(\boldsymbol{v}(\boldsymbol{x},\tau) + \sum_{i=1}^3 \boldsymbol{W}^i(\boldsymbol{y}, t-\tau) \otimes \boldsymbol{e}_i \cdot \tilde{\boldsymbol{z}}(\boldsymbol{x},\tau)\Big)\, d\tau,$$

$$R = \int_0^t \sum_{i=1}^3 R^i(\boldsymbol{y}, t-\tau)\boldsymbol{e}_i \cdot \tilde{\boldsymbol{z}}(\boldsymbol{x},\tau)\, d\tau,$$

where the functions $\boldsymbol{W}^i(\boldsymbol{y},t)$ and $R^i(\boldsymbol{y},t)$ are determined by solving the periodic initial-boundary-value problems

$$\begin{gathered} \rho_s \frac{\partial^2 \boldsymbol{W}^i}{\partial t^2} - \lambda_1 \Delta \boldsymbol{W}^i + \nabla R^i = 0, \qquad \operatorname{div}_y \boldsymbol{W}^i = 0, \qquad \boldsymbol{y} \in Y_s, \quad t > 0, \\ \boldsymbol{W}^i = 0, \qquad \boldsymbol{y} \in \gamma, \quad t > 0, \\ \boldsymbol{W}^i(y,0) = 0, \qquad \rho_s \frac{\partial \boldsymbol{W}^i}{\partial t(y,0)} = \boldsymbol{e}_i, \qquad \boldsymbol{y} \in Y_s, \end{gathered} \tag{3.28}$$

$\boldsymbol{e}_i$ is the unit vector of the Cartesian coordinate system. Hence,

$$\boldsymbol{B}_1^s(t) = \sum_{i=1}^3 \Big\langle \frac{\partial \boldsymbol{W}^i}{\partial t} \Big\rangle_{Y_s} \otimes \boldsymbol{e}_i(t).$$

We note that, by virtue of the constraints imposed on the geometry of the elementary cell Y_s, problem (3.28) has a unique solution which is only generalized because of the unmatched initial and boundary conditions. Therefore, at $t = 0$, the function $\boldsymbol{B}_1^s(t)$ is nondifferentiable.

2. Let $\lambda_1 = 0$. Then, to solve system (3.6), (3.12) and (3.13), one first needs to determine the pressure $R(\boldsymbol{x}, t, \boldsymbol{y})$ by solving the Neumann problem for the Laplace equation in the domain Y_s:

$$R(\boldsymbol{x}, t, \boldsymbol{y}) = \sum_{i=1}^3 R_i(\boldsymbol{y})\boldsymbol{e}_i \cdot \tilde{\boldsymbol{z}}(\boldsymbol{x}, t),$$

where $R_i(\boldsymbol{y})$ is a periodic solution of the problem

$$\Delta_y R_i = 0, \quad \boldsymbol{y} \in Y_s, \qquad \nabla_y R_i \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{e}_i, \quad \boldsymbol{y} \in \gamma.$$

This problem has a unique (to within an arbitrary constant) solution. Formula (3.26) is the result of averaging of Eq. (3.12) and the equation

$$\boldsymbol{B}_2^s = \sum_{i=1}^3 \langle \nabla R_i(\boldsymbol{y}) \rangle_{Y_s} \otimes \boldsymbol{e}_i,$$

where the matrix $(1-m)I - \boldsymbol{B}_2^s$ is strictly positive definite. Indeed, let for an arbitrary unit vector $\boldsymbol{\xi}$,

$$\tilde{R} = \sum_{i=1}^3 R_i \xi_i.$$

Then,

$$(\boldsymbol{B}\boldsymbol{\xi})\boldsymbol{\xi} = \langle (\boldsymbol{\xi} - \nabla \tilde{R})^2 \rangle_{Y_f} > 0.$$

Lemma 11. *For all $(\boldsymbol{x}, t) \in \Omega_T$, the weak and strong limits θ, p_f, and p_s satisfy the averaged heat-conduction equation*

$$\hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*} \frac{\partial p_f}{\partial t} - \frac{\beta_{0s}}{\eta_0} \frac{\partial p_s}{\partial t} + (\beta_{0f} - \beta_{0s})(a_3^f + a_4^f) \Big\langle \frac{\partial \theta}{\partial t} \Big\rangle_\Omega = \operatorname{div} (\boldsymbol{B}^\theta \nabla \theta) + \Psi, \tag{3.29}$$

where the symmetric strictly positive definite matrix $\boldsymbol{B}^\theta$ is defined below.

Proof. The averaged heat-conduction equation (3.29) is the macroscopic heat-conduction equation (3.15) in which the expression $\langle \tilde{\varkappa}_0 \nabla_y \Theta \rangle_Y$ is replaced by the expression

$$\langle \tilde{\varkappa}_0 \nabla_y \Theta \rangle_Y = \boldsymbol{B}_0^\theta \cdot \nabla \theta.$$

The last formula is the result of solution of the microscopic heat-conduction equation (3.14) in the form

$$\Theta(\boldsymbol{x}, t, \boldsymbol{y}) = \sum_{i=1}^{3} \Theta_i(\boldsymbol{y}) \frac{\partial \theta}{\partial x_i}(\boldsymbol{x}, t),$$

where Θ_i $(i = 1, 2, 3)$ are periodic solutions of the equation $\operatorname{div}_y [\tilde{\varkappa}_0 (\nabla_y \Theta_i + \boldsymbol{e}_i)] = 0$ in the domain Y. In this case,

$$\boldsymbol{B}^\theta = \hat{\varkappa}_0 \boldsymbol{I} + \boldsymbol{B}_0^\theta, \qquad \boldsymbol{B}_0^\theta = \sum_{i=1}^{3} \nabla_y \langle \Theta_i \rangle_Y \otimes \boldsymbol{e}_i.$$

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