# Nonisothermal Filtration and Seismic Acoustics in Porous Soil: Thermoviscoelastic Equations and Lamé Equations

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**Abstract**—We consider a linear system of differential equations that describes the joint motion of an incompressible elastic porous body and an incompressible fluid filling the pores. The model is very complicated because the main differential equations contain the derivatives of expressions with nondifferentiable rapidly oscillating small and large coefficients. On the basis of Nguetseng's two-scale convergence method, we derive homogenized equations in a rigorous way; depending on the geometry of pores, these are either the thermoviscoelasticity equations (for a connected porous space) or the anisotropic thermoelastic Lamé system.

## INTRODUCTION

In this paper we study the problem of joint motion of a nonisothermal elastic body perforated by a system of pores and channels (called a solid *skeleton*) and a nonisothermal viscous fluid filling the cavities (the *porous space*). We study only the case of a geometrically periodic porous space. Namely, we consider the domain $\Omega = (0,1)^3$ and assume that it is periodically composed of copies of an elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$. The number $1/\varepsilon$ is integer, so that $\Omega$ contains an integer number of elementary cells. Let $Y_{\mathrm{s}}$ be the "solid phase" of the cell $Y$, and let the "fluid phase" $Y_{\mathrm{f}}$ be its open complement. We set $\gamma = \partial Y_{\mathrm{f}} \cap \partial Y_{\mathrm{s}}$. The boundary $\gamma$ is a $C^1$ surface, the porous space $\Omega_{\mathrm{f}}^\varepsilon$ is the periodic repetition of the elementary cell $\varepsilon Y_{\mathrm{f}}$, the solid skeleton $\Omega_{\mathrm{s}}^\varepsilon$ is the periodic repetition of the elementary cell $\varepsilon Y_{\mathrm{s}}$, and the boundary $\Gamma^\varepsilon = \partial\Omega_{\mathrm{s}}^\varepsilon \cap \partial\Omega_{\mathrm{f}}^\varepsilon$ is the periodic repetition of the boundary $\varepsilon\gamma$ in $\Omega$.

In dimensionless variables, the differential equations of the problem for the dimensionless displacement $\mathbf{w}$ and the dimensionless temperature $\theta$ in the domain $\Omega \in \mathbb{R}^3$ have the form

$$\alpha_\tau \rho^\varepsilon \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname{div} \mathbb{P} + \rho^\varepsilon \mathbf{F}, \tag{1}$$

$$\alpha_\tau c_p^\varepsilon \frac{\partial \theta}{\partial t} = \operatorname{div}_x(\alpha_\varkappa^\varepsilon \nabla_x \theta) - \alpha_\theta^\varepsilon \frac{\partial}{\partial t} \operatorname{div}_x \mathbf{w}, \tag{2}$$

$$p + \alpha_p \chi^\varepsilon \operatorname{div}_x \mathbf{w} = 0, \tag{3}$$

$$\pi + \alpha_\eta (1 - \chi^\varepsilon) \operatorname{div}_x \mathbf{w} = 0, \tag{4}$$

$$\mathbb{P} = \chi^\varepsilon \alpha_\mu \mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) + (1 - \chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, \mathbf{w}) - (p + \pi - \alpha_\theta^\varepsilon \theta)\mathbb{I}.$$

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Here and in what follows, we use the notation

$$\mathbb{D}(x,\mathbf{u})=\frac{1}{2}\big(\nabla_x\mathbf{u}+(\nabla_x\mathbf{u})^T\big),\qquad \chi^\varepsilon(\mathbf{x})=\chi\Big(\frac{\mathbf{x}}{\varepsilon}\Big),$$

$$\rho^\varepsilon(\mathbf{x})=\chi^\varepsilon(\mathbf{x})\rho_{\mathrm{f}}+(1-\chi^\varepsilon(\mathbf{x}))\rho_{\mathrm{s}},\qquad c_p^\varepsilon(\mathbf{x})=\chi^\varepsilon(\mathbf{x})c_{p\mathrm{f}}+(1-\chi^\varepsilon(\mathbf{x}))c_{p\mathrm{s}},$$

$$\alpha_\varkappa^\varepsilon(\mathbf{x})=\chi^\varepsilon(\mathbf{x})\alpha_{\varkappa\mathrm{f}}+(1-\chi^\varepsilon(\mathbf{x}))\alpha_{\varkappa\mathrm{s}},\qquad \alpha_\theta^\varepsilon(\mathbf{x})=\chi^\varepsilon(\mathbf{x})\alpha_{\theta\mathrm{f}}+(1-\chi^\varepsilon(\mathbf{x}))\alpha_{\theta\mathrm{s}};$$

by $\chi(\mathbf{y})$ we denote the characteristic function of $Y_{\mathrm{f}}$ on $Y$. This function, which determines the geometry of the porous space, is assumed to be known in our model.

The derivation of equations (1)–(4) and the description of the dimensionless constants (which are all strictly positive) are given in [1].

The problem is supplemented by the homogeneous initial and boundary conditions.

In this model, a natural small parameter is the ratio $\varepsilon=l/L$ of the average pore size $l$ to the characteristic size $L$ of the domain under consideration; the dimensionless parameters $\alpha_i$, where $i=\tau,\nu,\dots$, depend on the small parameter $\varepsilon$. We also assume that the following (finite or infinite) limits exist:

$$\lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon)=\mu_0,\qquad \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon)=\lambda_0,\qquad \lim_{\varepsilon\searrow 0}\alpha_\tau(\varepsilon)=\tau_0,\qquad \lim_{\varepsilon\searrow 0}\alpha_p(\varepsilon)=p_*,\qquad \lim_{\varepsilon\searrow 0}\alpha_\eta(\varepsilon)=\eta_0,$$

$$\lim_{\varepsilon\searrow 0}\alpha_{\varkappa\mathrm{s}}(\varepsilon)=\varkappa_{0\mathrm{s}},\qquad \lim_{\varepsilon\searrow 0}\alpha_{\varkappa\mathrm{f}}=\varkappa_{0\mathrm{f}},\qquad \lim_{\varepsilon\searrow 0}\alpha_{\theta\mathrm{f}}(\varepsilon)=\beta_{0\mathrm{f}},\qquad \lim_{\varepsilon\searrow 0}\alpha_{\theta\mathrm{s}}(\varepsilon)=\beta_{0\mathrm{s}}.$$

Our main purpose in this paper is to find limit regimes (homogenized equations) as the small parameter tends to zero.

Simpler models for isothermal media were studied in [2–8].

## 1. STATEMENT OF THE MAIN RESULTS

As usual, equations (1) and (2) are understood in the sense of distribution theory. They include equations (1) and (2) on each of the domains $\Omega_{\mathrm{f}}^\varepsilon$ and $\Omega_{\mathrm{s}}^\varepsilon$ and the boundary conditions

$$[\vartheta]=0,\qquad [\mathbf{w}]=0,\qquad \mathbf{x}_0\in\Gamma^\varepsilon,\quad t\geq 0, \tag{1.1}$$

$$[\mathbb{P}\cdot\mathbf{n}]=0,\qquad [\alpha_\varkappa^\varepsilon\nabla_x\theta\cdot\mathbf{n}]=0,\qquad \mathbf{x}_0\in\Gamma^\varepsilon,\quad t\geq 0, \tag{1.2}$$

on the boundary $\Gamma^\varepsilon$, where $\mathbf{n}$ is the unit normal vector to the boundary and

$$[\varphi](\mathbf{x}_0)=\varphi_{(\mathrm{s})}(\mathbf{x}_0)-\varphi_{(\mathrm{f})}(\mathbf{x}_0),\qquad \varphi_{(\mathrm{s})}(\mathbf{x}_0)=\lim_{\substack{\mathbf{x}\to\mathbf{x}_0\\ \mathbf{x}\in\Omega_{\mathrm{s}}^\varepsilon}}\varphi(\mathbf{x}),\qquad \varphi_{(\mathrm{f})}(\mathbf{x}_0)=\lim_{\substack{\mathbf{x}\to\mathbf{x}_0\\ \mathbf{x}\in\Omega_{\mathrm{f}}^\varepsilon}}\varphi(\mathbf{x}).$$

There exist various equivalent (in the sense of distribution theory) forms of equations (1), (2) and the boundary conditions (1.1), (1.2). The form most convenient for our purposes is that of integral identities.

**Definition 1.** Functions $(\mathbf{w}^\varepsilon,\theta^\varepsilon,p^\varepsilon,\pi^\varepsilon)$ are called a generalized solution of problem (1)–(4), (1.1), (1.2) if they satisfy the regularity conditions

$$\mathbf{w}^\varepsilon,\ \mathbb{D}(x,\mathbf{w}^\varepsilon),\ \operatorname{div}_x\mathbf{w}^\varepsilon,\ p^\varepsilon,\ \theta^\varepsilon,\nabla_x\theta^\varepsilon\in L^2(\Omega_T) \tag{1.3}$$

in the domain $\Omega_T=\Omega\times(0,T)$, the boundary conditions

$$\mathbf{w}^\varepsilon=0,\qquad \theta^\varepsilon=0,\qquad \mathbf{x}\in S=\partial\Omega,\quad t\geq 0, \tag{1.4}$$

the continuity equations

$$\frac{1}{\alpha_p}p^\varepsilon+\chi^\varepsilon\operatorname{div}_x\mathbf{w}^\varepsilon=-\frac{1}{m}\gamma^\varepsilon\chi^\varepsilon \tag{1.5}$$

and

$$\frac{1}{\alpha_p}\pi^\varepsilon + (1-\chi^\varepsilon)\operatorname{div}_x \mathbf{w}^\varepsilon = \frac{1}{1-m}\gamma^\varepsilon(1-\chi^\varepsilon) \tag{1.6}$$

almost everywhere in the domain $\Omega_T$, the integral identity

$$\int\limits_{\Omega_T}\Big(\alpha_\tau\rho^\varepsilon\mathbf{w}^\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2} - \rho^\varepsilon\mathbf{F}\cdot\varphi - \chi^\varepsilon\alpha_\mu\mathbb{D}(\mathbf{x},\mathbf{w}^\varepsilon):\mathbb{D}\Big(x,\frac{\partial\varphi}{\partial t}\Big)$$
$$+\big\{(1-\chi^\varepsilon)\alpha_\lambda\mathbb{D}(x,\mathbf{w}^\varepsilon) - \big(p^\varepsilon+\pi^\varepsilon-\alpha_\theta^\varepsilon\theta^\varepsilon\big)\mathbb{I}\big\}:\mathbb{D}(x,\varphi)\Big)\,d\mathbf{x}\,dt = 0 \tag{1.7}$$

for all smooth vector functions $\varphi=\varphi(\mathbf{x},t)$ such that $\varphi|_{\partial\Omega}=\varphi|_{t=T}=\frac{\partial\varphi}{\partial t}\big|_{t=T}=0$, and the integral identity

$$\int\limits_{\Omega_T}\Big(\big(\alpha_\tau c_p^\varepsilon\theta^\varepsilon+\alpha_\theta^\varepsilon\operatorname{div}_x\mathbf{w}^\varepsilon\big)\frac{\partial\xi}{\partial t} - \alpha_\varkappa^\varepsilon\nabla_x\theta^\varepsilon\cdot\nabla_x\xi\Big)\,d\mathbf{x}\,dt = 0 \tag{1.8}$$

for all smooth functions $\xi=\xi(\mathbf{x},t)$ such that $\xi|_{\partial\Omega}=\xi|_{t=T}=0$.

In (1.7), $A:B$ denotes the convolution of two rank 2 tensors over both indices; i.e., $A:B=\operatorname{tr}(B^*\circ A)=\sum_{i,j=1}^3 A_{ij}B_{ji}$. The normalizing term $\gamma^\varepsilon=\int_\Omega(1-\chi^\varepsilon)\operatorname{div}_x\mathbf{w}^\varepsilon\,d\mathbf{x}$ in (1.5) and (1.6) is chosen so that

$$\int\limits_\Omega p^\varepsilon\,d\mathbf{x} = \int\limits_\Omega \pi^\varepsilon\,d\mathbf{x} = 0.$$

In what follows, we assume that the following holds.

**Assumption 1.** 1. The dimensionless parameters in problem (1)–(4), (1.1), (1.2) satisfy the restrictions

$$0<\varkappa_{0\mathrm{f}},\varkappa_{0\mathrm{s}},\mu_0,\lambda_0<\infty,\qquad \tau_0,\beta_{0\mathrm{f}},\beta_{0\mathrm{s}},p_*^{-1},\eta_0^{-1}<\infty,\qquad p_*+\eta_0=\infty.$$

2. The functions $\mathbf{F}$, $\partial\mathbf{F}/\partial t$, and $\partial^2\mathbf{F}/\partial t^2$ are bounded in $L^2(\Omega_T)$.

The condition $p_*=\infty$ means that the fluid under consideration is incompressible, and the condition $\eta_0=\infty$ means that the solid skeleton is incompressible. Throughout the paper, the model parameters are allowed to take any admissible values. For example, if $\tau_0=0$ or $\eta_0^{-1}=0$, then the terms containing these quantities vanish in all equations.

The main results of this paper are formulated in Theorems 1 and 2.

**Theorem 1.** *Under the assumptions made above, for any $\varepsilon>0$, there exists a unique generalized solution of problem* (1)–(4), (1.1), (1.2) *on an arbitrary time interval* $[0,T]$, *and the following estimates hold*:

$$\max_{0\le t\le T}\Big\|\Big|\frac{\partial\mathbf{w}^\varepsilon}{\partial t}(t)\Big| + \alpha_\tau\Big|\frac{\partial^2\mathbf{w}^\varepsilon}{\partial t^2}(t)\Big| + \Big|\nabla_x\frac{\partial\mathbf{w}^\varepsilon}{\partial t}(t)\Big|\Big\|_{2,\Omega}\le C_0, \tag{1.9}$$

$$\max_{0\le t\le T}\Big(\Big\|\frac{\partial\theta^\varepsilon}{\partial t}\Big\|_{2,\Omega}+\|\nabla_x\theta^\varepsilon\|_{2,\Omega}\Big)\le C_0, \tag{1.10}$$

$$\max_{0\le t\le T}\big(\|\pi^\varepsilon(t)\|_{2,\Omega}+\|p^\varepsilon(t)\|_{2,\Omega}\big)\le C_0, \tag{1.11}$$

*where the constant $C_0$ does not depend on the small parameter $\varepsilon$.*

**Theorem 2.** *The sequences $\{\mathbf{w}^\varepsilon\}$ and $\{\theta^\varepsilon\}$ converge strongly in $L^2(\Omega_T)$ and weakly in $L^2((0,T);W_2^1(\Omega))$ to functions $\mathbf{w}$ and $\theta$, and the sequences $\{p^\varepsilon\}$ and $\{\pi^\varepsilon\}$ converge weakly in $L^2(\Omega_T)$ to functions $p$ and $\pi$, respectively. Moreover, the functions $\mathbf{w}$, $\theta$, $p$, and $\pi$ satisfy the following initial–boundary value problem in $\Omega_T$:*

$$\tau_0\widehat{\rho}\frac{\partial^2\mathbf{w}}{\partial t^2}+\nabla(q+\pi-\widehat{\beta}_0\theta)-\widehat{\rho}\mathbf{F}=\operatorname{div}_x\Bigg(\mathbb{A}_1:\mathbb{D}\Big(x,\frac{\partial\mathbf{w}}{\partial t}\Big)+\mathbb{A}_2:\mathbb{D}(x,\mathbf{w})$$
$$+\int_0^t\Big(\mathbb{A}_3(t-\tau):\mathbb{D}(x,\mathbf{w}(\mathbf{x},\tau))+B(t-\tau)\operatorname{div}_x\mathbf{w}(\mathbf{x},\tau)+B^\theta(t-\tau)\theta(\mathbf{x},\tau)\Big)\,d\tau\Bigg),\quad(1.12)$$

$$\frac{1}{p_*}p+m\operatorname{div}_x\mathbf{w}=-\int_0^t\Big(C_1(t-\tau):\mathbb{D}(x,\mathbf{w}(\mathbf{x},\tau))+a_1(t-\tau)\operatorname{div}_x\mathbf{w}(\mathbf{x},\tau)$$
$$+a_1^\theta(t-\tau)\theta(\mathbf{x},\tau)\Big)\,d\tau-a(t)\langle\theta\rangle_\Omega,\quad(1.13)$$

$$\frac{1}{\eta_0}\pi+(1-m)\operatorname{div}_x\mathbf{w}=-\int_0^t\Big(C_2(t-\tau):\mathbb{D}(x,\mathbf{w}(\mathbf{x},\tau))+a_2(t-\tau)\operatorname{div}_x\mathbf{w}(\mathbf{x},\tau)$$
$$+a_2^\theta(t-\tau)\theta(\mathbf{x},\tau)\Big)\,d\tau+a(t)\langle\theta\rangle_\Omega,\quad(1.14)$$

$$\tau_0\widehat{c}_p\frac{\partial\theta}{\partial t}-\frac{\beta_{0\mathrm{f}}}{p_*}\frac{\partial p}{\partial t}-\frac{\beta_{0\mathrm{s}}}{\eta_0}\frac{\partial\pi}{\partial t}+(\beta_{0\mathrm{s}}-\beta_{0\mathrm{f}})\frac{\partial}{\partial t}(a(t)\langle\theta\rangle_\Omega)=\operatorname{div}_x\big(B_0^\theta\cdot\nabla_x\theta\big)+\Psi.\quad(1.15)$$

*The problem is supplemented by the homogeneous initial and boundary conditions for $\theta$ and $\mathbf{w}$.*

*In* (1.12)–(1.15),

$$\widehat{\rho}=\rho_{\mathrm{f}}m+\rho_{\mathrm{s}}(1-m),\qquad\widehat{\beta}_0=\beta_{0\mathrm{f}}m+\beta_{0\mathrm{s}}(1-m),\qquad\widehat{c}_p=c_{p\mathrm{f}}m+c_{p\mathrm{s}}(1-m),\qquad m=\int_Y\chi(y)\,dy;$$

*$\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{A}_3(t)$ are rank 4 tensors; $B(t)$, $B^\theta(t)$, $B_0^\theta$, $C_1(t)$, and $C_2(t)$ are matrices; and $a_1(t)$, $a_2(t)$, $a_1^\theta(t)$, $a_2^\theta(t)$, and $a(t)$ are scalars. Exact expressions for these objects are given below in formulas* (4.17)–(4.24). *The matrix $B_0^\theta$ is strictly positive definite.*

*For a connected porous space, the symmetric tensor $\mathbb{A}_1$ is strictly positive definite. Otherwise (for a disconnected porous space, when $\gamma\cap\partial Y=\varnothing$), $\mathbb{A}_1=0$ and system* (1.12) *degenerates into a nonlocal anisotropic system of Lamé equations with a symmetric strictly positive definite tensor $\mathbb{A}_2$.*

## 2. PRELIMINARIES

The proof of Theorem 2 is based on a systematic application of the two-scale convergence method proposed by Nguetseng [9], which has been extensively used in homogenization theory (see, e.g., the survey [10] and Zhikov's papers [11–13]).

**Definition 2.** A sequence $\{\varphi^\varepsilon\}\subset L^2(\Omega_T)$ *two-scale converges* to a limit $\varphi\in L^2(\Omega_T\times Y)$ if

$$\lim_{\varepsilon\searrow 0}\int_{\Omega_T}\varphi^\varepsilon(\mathbf{x},t)\sigma(\mathbf{x},t,\mathbf{x}/\varepsilon)\,d\mathbf{x}\,dt=\int_{\Omega_T}\int_Y\varphi(\mathbf{x},t,\mathbf{y})\sigma(\mathbf{x},t,\mathbf{y})\,d\mathbf{y}\,d\mathbf{x}\,dt\quad(2.1)$$

for any smooth function $\sigma=\sigma(\mathbf{x},t,\mathbf{y})$ that is 1-periodic in $\mathbf{y}$.

The existence and basic properties of two-scale convergent sequences are stated in the following theorem [9, 10].

**Theorem 3** (Nguetseng's theorem). 1. *Any bounded sequence in* $L^2(\Omega_T)$ *contains a subsequence that two-scale converges to some limit* $\varphi \in L^2(\Omega_T \times Y)$.

2. *Suppose that sequences* $\{\varphi^\varepsilon\}$ *and* $\{\varepsilon\nabla_x\varphi^\varepsilon\}$ *are uniformly bounded in* $L^2(\Omega_T)$. *Then there exist a function* $\varphi = \varphi(\mathbf{x}, t, \mathbf{y})$ 1*-periodic in* $\mathbf{y}$ *and a subsequence of* $\{\varphi^\varepsilon\}$ *such that* $\varphi, \nabla_y\varphi \in L^2(\Omega_T \times Y)$ *and* $\varphi^\varepsilon$ *and* $\varepsilon\nabla_x\varphi^\varepsilon$ *two-scale converge to* $\varphi$ *and* $\nabla_y\varphi$, *respectively.*

3. *Suppose that sequences* $\{\varphi^\varepsilon\}$ *and* $\{\nabla_x\varphi^\varepsilon\}$ *are uniformly bounded in* $L^2(\Omega_T)$. *Then there exist functions* $\varphi \in L^2(\Omega_T)$ *and* $\psi \in L^2(\Omega_T \times Y)$ *and a subsequence of* $\{\varphi^\varepsilon\}$ *such that* $\psi$ *is* 1*-periodic in* $\mathbf{y}$, $\nabla_y\psi \in L^2(\Omega_T \times Y)$, *and* $\varphi^\varepsilon$ *and* $\nabla_x\varphi^\varepsilon$ *two-scale converge to* $\varphi$ *and* $\nabla_x\varphi(\mathbf{x}, t) + \nabla_y\psi(\mathbf{x}, t, \mathbf{y})$, *respectively.*

**Corollary 1.** *Let* $\sigma \in L^2(Y)$ *and* $\sigma^\varepsilon(\mathbf{x}) := \sigma(\mathbf{x}/\varepsilon)$. *Suppose that a sequence* $\{\varphi^\varepsilon\} \subset L^2(\Omega_T)$ *two-scale converges to a limit* $\varphi \in L^2(\Omega_T \times Y)$. *Then, the sequence of* $\sigma^\varepsilon\varphi^\varepsilon$ *two-scale converges to* $\sigma\varphi$.

Throughout the paper, we use the following notation:

(1) $\langle\Phi\rangle_Y = \int_Y \Phi\, d\mathbf{y}$, $\langle\Phi\rangle_{Y_\mathrm{f}} = \int_Y \chi\Phi\, d\mathbf{y}$, $\langle\Phi\rangle_{Y_\mathrm{s}} = \int_Y (1-\chi)\Phi\, d\mathbf{y}$, $\langle\varphi\rangle_\Omega = \int_\Omega \varphi\, d\mathbf{x}$, and $\langle\varphi\rangle_{\Omega_T} = \int_{\Omega_T} \varphi\, d\mathbf{x}\, dt$;

(2) if $\mathbf{a}$ and $\mathbf{b}$ are vectors, then $\mathbf{a} \otimes \mathbf{b}$ is the matrix defined by

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

for any vector $\mathbf{c}$;

(3) if $B$ and $C$ are matrices, then $B \otimes C$ is a rank 4 tensor such that its convolution with any matrix $A$ is given by

$$(B \otimes C) : A = B(C : A);$$

(4) $\mathbb{I}^{ij}$ is the matrix with a unique nonzero element that equals 1 and is in the intersection of the $i$th row and the $j$th column;

(5) finally,

$$J^{ij} = \frac{1}{2}(\mathbb{I}^{ij} + \mathbb{I}^{ji}) = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i),$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis.

## 3. PROOF OF THEOREM 1

For all $\varepsilon > 0$, we have

$$\max_{0<t<T}\left(\sqrt{\alpha_\lambda}\left\|\nabla_x \frac{\partial \mathbf{w}^\varepsilon}{\partial t}(t)\right\|_{2,\Omega^\varepsilon_\mathrm{s}} + \sqrt{\alpha_\tau}\left\|\frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}(t)\right\|_{2,\Omega} + \sqrt{\alpha_p}\left\|\mathrm{div}_x \frac{\partial \mathbf{w}^\varepsilon}{\partial t}(t)\right\|_{2,\Omega^\varepsilon_\mathrm{f}} + \sqrt{\alpha_\tau}\left\|\frac{\partial \theta^\varepsilon}{\partial t}(t)\right\|_{2,\Omega}\right)$$
$$+ \sqrt{\alpha_\mu}\left\|\chi^\varepsilon\nabla_x \frac{\partial^2 \mathbf{w}^\varepsilon}{\partial t^2}\right\|_{2,\Omega_T} + \left\|\sqrt{\alpha^\varepsilon_\theta}\nabla_x \frac{\partial \theta^\varepsilon}{\partial t}\right\|_{2,\Omega_T} \leq C_0, \quad (3.1)$$

where $C_0$ does not depend on $\varepsilon$. This estimate is obtained by differentiating the equations for $\mathbf{w}^\varepsilon$ and $\theta^\varepsilon$ with respect to time, multiplying the first equation by $\partial^2\mathbf{w}^\varepsilon/\partial t^2$ and the second by $\partial\theta^\varepsilon/\partial t$, integrating by parts, and summing. This estimate ensures the existence and uniqueness of a generalized solution to problem (1)–(4), (1.1), (1.2).

Estimates (1.9) and (1.10) follow from estimates (3.1) and the Poincaré inequality.

Estimate (1.11) for pressures is derived as an estimate for the corresponding functional from the integral identity (1.7) and inequalities (1.9) and (1.10) in view of the relation

$$\int\limits_{\Omega} \big(p^{\varepsilon}(\mathbf{x},t) + \pi^{\varepsilon}(\mathbf{x},t)\big)\, d\mathbf{x} = 0.$$

Indeed, the integral identity (1.7) implies

$$\left| \int\limits_{\Omega} \big(p^{\varepsilon}(\mathbf{x},t) + \pi^{\varepsilon}(\mathbf{x},t)\big) \operatorname{div}_x \boldsymbol{\psi}\, d\mathbf{x} \right| \leq C \|\nabla \boldsymbol{\psi}\|_{2,\Omega}.$$

Choosing $\boldsymbol{\psi}$ so that $p^{\varepsilon} + \pi^{\varepsilon} = \operatorname{div}_x \boldsymbol{\psi}$, we obtain an estimate for the sum of pressures $p^{\varepsilon} + \pi^{\varepsilon}$. Such a choice is always possible (see [14]); it suffices to set

$$\boldsymbol{\psi} = \nabla \varphi + \boldsymbol{\psi}_0, \qquad \text{where} \quad \operatorname{div}_x \boldsymbol{\psi}_0 = 0, \quad \Delta \varphi = p^{\varepsilon} + \pi^{\varepsilon}, \quad \varphi|_{\partial\Omega} = 0, \quad \text{and} \quad (\nabla\varphi + \boldsymbol{\psi}_0)|_{\partial\Omega} = 0.$$

It remains to note that since the functions $p^{\varepsilon}$ and $\pi^{\varepsilon}$ are orthogonal, a bound for the sum implies a bound for each term.

## 4. PROOF OF THEOREM 2

**4.1. Weak and two-scale limits of the sequences of displacements, temperatures, and pressures.** By Theorem 1, the sequences $\{\mathbf{w}^{\varepsilon}\}$, $\{\theta^{\varepsilon}\}$, $\{p^{\varepsilon}\}$, and $\{\pi^{\varepsilon}\}$ are uniformly bounded with respect to $\varepsilon$ in $L^2(\Omega_T)$. Therefore, there exist a subsequence of $\varepsilon > 0$ and functions $p$, $\pi$, $\mathbf{w}$, and $\theta$ such that, as $\varepsilon \searrow 0$,

$$p^{\varepsilon} \to p, \qquad \pi^{\varepsilon} \to \pi \qquad \text{weakly in } L^2(\Omega_T),$$

$$\mathbf{w}^{\varepsilon} \to \mathbf{w}, \qquad \theta^{\varepsilon} \to \theta \qquad \text{strongly in } L^2(\Omega_T),$$

$$\nabla_x \mathbf{w}^{\varepsilon} \to \nabla_x \mathbf{w}, \qquad \nabla_x \theta^{\varepsilon} \to \nabla_x \theta \qquad \text{weakly in } L^2(\Omega_T).$$

These relations and Nguetseng's theorem imply the existence of functions $P(\mathbf{x},t,\mathbf{y})$, $\Pi(\mathbf{x},t,\mathbf{y})$, $\Theta(\mathbf{x},t,\mathbf{y})$, and $\mathbf{W}(\mathbf{x},t,\mathbf{y})$ that are 1-periodic in $\mathbf{y}$ and for which the sequences $\{p^{\varepsilon}\}$, $\{\pi^{\varepsilon}\}$, $\{\nabla\theta^{\varepsilon}\}$, and $\{\nabla\mathbf{w}^{\varepsilon}\}$ two-scale converge as $\varepsilon \searrow 0$ to $P$, $\Pi$, $\nabla_x\theta + \nabla_y\Theta$, and $\nabla_x\mathbf{w} + \nabla_y\mathbf{W}$, respectively.

**4.2. Micro- and macroscopic equations.**

**Lemma 4.1.** *The two-scale limits of the sequences* $\{p^{\varepsilon}\}$, $\{\pi^{\varepsilon}\}$, $\{\nabla\theta^{\varepsilon}\}$, *and* $\{\nabla\mathbf{w}^{\varepsilon}\}$ *satisfy the following microscopic relations in* $Y_T = Y \times (0,T)$:

$$\frac{1}{\eta_0}\Pi + (1-\chi)\big(\operatorname{div}_x \mathbf{w} + \operatorname{div}_y \mathbf{W}\big) = \frac{\gamma}{(1-m)}(1-\chi), \tag{4.1}$$

$$\frac{1}{p_*}P + \chi\big(\operatorname{div}_x \mathbf{w} + \operatorname{div}_y \mathbf{W}\big) = -\frac{\gamma}{m}\chi, \tag{4.2}$$

$$\begin{aligned} \nabla_y\big(P + \Pi - (\beta_{0\mathrm{f}}\chi + \beta_{0\mathrm{s}}(1-\chi))\theta\big) = \operatorname{div}_y \bigg(\chi\mu_0\Big(\mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) + \mathbb{D}\Big(y, \frac{\partial \mathbf{W}}{\partial t}\Big)\Big) \\ + (1-\chi)\lambda_0\big(\mathbb{D}(x,\mathbf{w}) + \mathbb{D}(y,\mathbf{W})\big)\bigg), \end{aligned} \tag{4.3}$$

$$\operatorname{div}_y\big(\chi\varkappa_{0\mathrm{f}}(\nabla_x\theta + \nabla_y\Theta) + (1-\chi)\varkappa_{0\mathrm{s}}(\nabla_x\theta + \nabla_y\Theta)\big) = 0. \tag{4.4}$$

**Lemma 4.2.** *The weak and strong limits $p$, $\pi$, $\theta$, and $\mathbf{w}$ satisfy the following system of macroscopic equations in $\Omega_T$:*

$$\frac{1}{\eta_0}\pi + (1-m)\operatorname{div}_x \mathbf{w} + \langle \operatorname{div}_y \mathbf{W}\rangle_{Y_{\mathrm{s}}} = \gamma, \tag{4.5}$$

$$\frac{1}{p_*}p + m\operatorname{div}_x \mathbf{w} + \langle \operatorname{div}_y \mathbf{W}\rangle_{Y_{\mathrm{f}}} = -\gamma, \tag{4.6}$$

$$\tau_0\widehat{\rho}\frac{\partial^2 \mathbf{w}}{\partial t^2} + \nabla(q+\pi-\widehat{\beta}_0\theta) - \widehat{\rho}\mathbf{F} = \operatorname{div}_x\bigg(\mu_0\bigg(m\mathbb{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big) + \Big\langle \mathbb{D}\Big(y,\frac{\partial \mathbf{W}}{\partial t}\Big)\Big\rangle_{Y_{\mathrm{f}}}\bigg)$$
$$+ \lambda_0\big((1-m)\mathbb{D}(x,\mathbf{w}) + \langle \mathbb{D}(y,\mathbf{W})\rangle_{Y_{\mathrm{f}}}\big)\bigg), \tag{4.7}$$

$$\tau_0\widehat{c}_p\frac{\partial \theta}{\partial t} - \frac{\beta_{0\mathrm{f}}}{p_*}\frac{\partial p}{\partial t} - \frac{\beta_{0\mathrm{s}}}{\eta_0}\frac{\partial \pi}{\partial t} + (\beta_{0\mathrm{s}}-\beta_{0\mathrm{f}})\frac{\partial \gamma}{\partial t} - \Psi = \operatorname{div}_x\Big(\varkappa_{0\mathrm{f}}\big(m\nabla_x\theta + \langle \nabla_y\Theta\rangle_{Y_{\mathrm{f}}}\big)$$
$$+ \varkappa_{0\mathrm{s}}\big((1-m)\nabla_x\theta + \langle \nabla_y\Theta\rangle_{Y_{\mathrm{s}}}\big)\Big). \tag{4.8}$$

In (4.1)–(4.8),

$$\gamma = \big\langle(\langle \operatorname{div}_y \mathbf{W}\rangle_{Y_{\mathrm{s}}})\big\rangle_{\Omega}, \qquad \widehat{\rho} = \rho_{\mathrm{f}}m + \rho_{\mathrm{s}}(1-m),$$
$$\widehat{\beta}_0 = \beta_{0\mathrm{f}}m + \beta_{0\mathrm{s}}(1-m), \qquad \widehat{c}_p = c_{p\mathrm{f}}m + c_{p\mathrm{s}}(1-m).$$

**Proof.** To prove (4.1) and (4.2), we multiply equations (1.5) and (1.6) by $\psi^\varepsilon = \psi(\mathbf{x},t,\mathbf{x}/\varepsilon)$, where $\psi(\mathbf{x},t,\mathbf{y})$ is an arbitrary function 1-periodic in $\mathbf{y}$, and integrate the result over the domain $\Omega$. The required relations are obtained by passing to the limit as $\varepsilon \searrow 0$.

Equations (4.3) and (4.4) follow from the integral identities (1.8) and (1.9) with the test functions $\boldsymbol{\varphi}^\varepsilon = \varepsilon\boldsymbol{\varphi}(\mathbf{x},t,\mathbf{x}/\varepsilon)$ (in (1.8)) and $\xi^\varepsilon = \varepsilon\xi(\mathbf{x},t,\mathbf{x}/\varepsilon)$ (in (1.9)), where $\boldsymbol{\varphi}(\mathbf{x},t,\mathbf{y})$ and $\xi(\mathbf{x},t,\mathbf{y})$ are arbitrary functions 1-periodic in $\mathbf{y}$, by passing to the limit as $\varepsilon \searrow 0$.

Equations (4.5) and (4.6) result from homogenizing equations (4.1) and (4.2) over the elementary cell $Y$, and equations (4.7) and (4.8) follow from the integral identities (1.8) and (1.9) with test functions independent of the "fast" variable $y$ by passing to the limit as $\varepsilon \searrow 0$ and applying the continuity equations (1.5) and (1.6) to identity (1.9). □

### 4.3. Homogenized equations.

**Lemma 4.3.** *The weak and strong limits $p$, $\pi$, $\theta$, and $\mathbf{w}$ satisfy the following system of homogenized equations in $\Omega_T$:*

$$\tau_0\widehat{\rho}\frac{\partial^2 \mathbf{w}}{\partial t^2} + \nabla(q+\pi-\widehat{\beta}_0\theta) - \widehat{\rho}\mathbf{F} = \operatorname{div}_x\bigg(\mathbb{A}_1 : \mathbb{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big) + \mathbb{A}_2 : \mathbb{D}(x,\mathbf{w})$$
$$+ \int_0^t \Big(\mathbb{A}_3(t-\tau) : \mathbb{D}(x,\mathbf{w}(\mathbf{x},\tau)) + B(t-\tau)\operatorname{div}_x \mathbf{w}(\mathbf{x},\tau) + B^\theta(t-\tau)\theta(\mathbf{x},\tau)\Big)\,d\tau\bigg), \tag{4.9}$$

$$\frac{1}{p_*}p + m\operatorname{div}_x \mathbf{w} = -\int_0^t \Big(C_1(t-\tau) : \mathbb{D}(x,\mathbf{w}(\mathbf{x},\tau)) + a_1(t-\tau)\operatorname{div}_x \mathbf{w}(\mathbf{x},\tau)$$
$$+ a_1^\theta(t-\tau)\theta(\mathbf{x},\tau)\Big)\,d\tau - a(t)\langle\theta\rangle_\Omega, \tag{4.10}$$

$$\frac{1}{\eta_0}\pi + (1-m)\operatorname{div}_x \mathbf{w} = -\int\limits_0^t \Big( C_2(t-\tau) : \mathbb{D}(x, \mathbf{w}(\mathbf{x},\tau)) + a_2(t-\tau)\operatorname{div}_x \mathbf{w}(\mathbf{x},\tau)$$

$$+ a_2^\theta(t-\tau)\theta(\mathbf{x},\tau)\Big)\, d\tau + a(t)\langle\theta\rangle_\Omega, \tag{4.11}$$

$$\tau_0 \widehat{c}_p \frac{\partial\theta}{\partial t} - \frac{\beta_{0\mathrm{f}}}{p_*}\frac{\partial p}{\partial t} - \frac{\beta_{0\mathrm{s}}}{\eta_0}\frac{\partial \pi}{\partial t} + (\beta_{0\mathrm{s}} - \beta_{0\mathrm{f}})\frac{\partial}{\partial t}(a(t)\langle\theta\rangle_\Omega) = \operatorname{div}_x\big(B_0^\theta \cdot \nabla_x \theta\big) + \Psi. \tag{4.12}$$

*Here* $\mathbb{A}_1$, $\mathbb{A}_2$, *and* $\mathbb{A}_3(t)$ *are rank 4 tensors*; $B(t)$, $B^\theta(t)$, $B_0^\theta$, $C_1(t)$, *and* $C_2(t)$ *are matrices*; *and* $a_1(t)$, $a_2(t)$, $a_1^\theta(t)$, $a_2^\theta(t)$, *and* $a(t)$ *are scalars. Exact expressions of these objects are given below* (*see* (4.17)–(4.24)).

**Proof.** We set

$$\mathbb{Z}(\mathbf{x},t) = \mu_0 \mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) - \lambda_0 \mathbb{D}(x,\mathbf{w}), \qquad Z_{ij} = \mathbf{e}_i \cdot (\mathbb{Z}\cdot \mathbf{e}_j), \qquad z_1(t) = \langle\theta\rangle_\Omega,$$

$$\mathbf{z}(\mathbf{x},t) = \sum_{i=1}^3 z_i(\mathbf{x},t)\mathbf{e}_i = (\varkappa_{0\mathrm{f}} - \varkappa_{0\mathrm{f}})\nabla_x\theta, \qquad z_0(\mathbf{x},t) = \operatorname{div}_x \mathbf{w}.$$

As usual, we seek solutions to the microscopic equations (4.1)–(4.4) in the form

$$\mathbf{W} = \int\limits_0^t \Big[ \mathbf{W}^0(\mathbf{y},t-\tau) z_0(\mathbf{x},\tau) + \sum_{i,j=1}^3 \mathbf{W}^{ij}(\mathbf{y},t-\tau) Z_{ij}(\mathbf{x},\tau)$$

$$+ \mathbf{W}^\theta(\mathbf{y},t-\tau)(\theta(\mathbf{x},\tau) - z_1(\tau)) + \mathbf{W}_1^\theta(\mathbf{y},t-\tau) z_1(\tau)\Big]\, d\tau, \tag{4.13}$$

$$P = \chi \int\limits_0^t \Big[ P^0(\mathbf{y},t-\tau) z_0(\mathbf{x},\tau) + \sum_{i,j=1}^3 P^{ij}(\mathbf{y},t-\tau) Z_{ij}(\mathbf{x},\tau)$$

$$+ P^\theta(\mathbf{y},t-\tau)(\theta(\mathbf{x},\tau) - z_1(\tau)) + P_1^\theta(\mathbf{y},t-\tau) z_1(\tau)\Big]\, d\tau, \tag{4.14}$$

$$\Pi = (1-\chi) \int\limits_0^t \Big[ \Pi^0(\mathbf{y},t-\tau) z_0(\mathbf{x},\tau) + \sum_{i,j=1}^3 \Pi^{ij}(\mathbf{y},t-\tau) Z_{ij}(\mathbf{x},\tau)$$

$$+ \Pi^\theta(\mathbf{y},t-\tau)(\theta(\mathbf{x},\tau) - z_1(\tau)) + \Pi_1^\theta(\mathbf{y},t-\tau) z_1(\tau)\Big]\, d\tau, \tag{4.15}$$

$$\Theta = \sum_{i=1}^3 \Theta^i(\mathbf{y}) z_i(\mathbf{x},t), \tag{4.16}$$

where $\mathbf{W}^0$, $\mathbf{W}^\theta$, $\mathbf{W}^{ij}$, $P^0$, $P^\theta$, $P^{ij}$, $\Pi^0$, $\Pi^\theta$, $\Pi^{ij}$, and $\Theta^i$ are functions 1-periodic in $\mathbf{y}$ and satisfying the following initial–boundary value problems on the elementary cell $Y$.

Problem (I):

$$\operatorname{div}_y\bigg( \chi\mu_0 \mathbb{D}\Big(y, \frac{\partial \mathbf{W}^{ij}}{\partial t}\Big) + (1-\chi)\big(\lambda_0 \mathbb{D}(y, \mathbf{W}^{ij}) - ((1-\chi)\Pi^{ij} + \chi P^{ij})\mathbb{I}\big)\bigg) = 0,$$

$$\frac{1}{p_*}P^{ij} + \chi \operatorname{div}_y \mathbf{W}^{ij} = 0, \qquad \frac{1}{\eta_0}\Pi^{ij} + (1-\chi)\operatorname{div}_y \mathbf{W}^{ij} = 0,$$

$$\mathbf{W}^{ij}(\mathbf{y},0) = \mathbf{W}_0^{ij}(\mathbf{y}), \qquad \operatorname{div}_y\big(\chi\big(\mu_0 \mathbb{D}(y, \mathbf{W}_0^{ij}) + J^{ij}\big)\big) = 0.$$

Problem (II):

$$\operatorname{div}_y\Big(\chi\mu_0\mathbb{D}\Big(y,\frac{\partial\mathbf{W}^0}{\partial t}\Big)+(1-\chi)\big(\lambda_0\mathbb{D}(y,\mathbf{W}^0)-((1-\chi)\Pi^0+\chi P^0)\mathbb{I}\big)\Big)=0,$$

$$\chi\mathbf{W}^0(\mathbf{y},0)=0,\qquad \frac{1}{p_*}P^0+\chi(\operatorname{div}_y\mathbf{W}^0+1)=0,\qquad \frac{1}{\eta_0}\Pi^0+(1-\chi)(\operatorname{div}_y\mathbf{W}^0+1)=0.$$

Problem (III):

$$\operatorname{div}_y\Big(\chi\mu_0\mathbb{D}\Big(y,\frac{\partial\mathbf{W}^\theta}{\partial t}\Big)+(1-\chi)\big(\lambda_0\mathbb{D}(y,\mathbf{W}^\theta)-\big((1-\chi)\Pi^\theta+\chi P^\theta-\beta_{0\mathrm{f}}\chi-\beta_{0\mathrm{s}}(1-\chi)\big)\mathbb{I}\big)\Big)=0,$$

$$\chi\mathbf{W}^\theta(\mathbf{y},0)=0,\qquad \frac{1}{p_*}P^\theta+\chi\operatorname{div}_y\mathbf{W}^\theta=0,\qquad \frac{1}{\eta_0}\Pi^\theta+(1-\chi)\operatorname{div}_y\mathbf{W}^\theta=0.$$

Problem (IV):

$$\operatorname{div}_y\Big(\chi\mu_0\mathbb{D}\Big(y,\frac{\partial\mathbf{W}_1^\theta}{\partial t}\Big)+(1-\chi)\big(\lambda_0\mathbb{D}(y,\mathbf{W}_1^\theta)-\big((1-\chi)\Pi_1^\theta+\chi P_1^\theta-\beta_{0\mathrm{f}}\chi-\beta_{0\mathrm{s}}(1-\chi)\big)\mathbb{I}\big)\Big)=0,$$

$$\mathbf{W}_1^\theta(\mathbf{y},0)=0,\qquad \frac{1}{p_*}P_1^\theta+\chi\operatorname{div}_y\mathbf{W}_1^\theta=-\frac{\chi}{m}\langle\operatorname{div}_y\mathbf{W}_1^\theta\rangle_{Y_\mathrm{s}},$$

$$\frac{1}{\eta_0}\Pi_1^\theta+(1-\chi)\operatorname{div}_y\mathbf{W}_1^\theta=\langle\operatorname{div}_y\mathbf{W}_1^\theta\rangle_{Y_\mathrm{s}}\frac{1-\chi}{1-m}.$$

Problem (V):

$$\operatorname{div}_y\big(\chi\varkappa_{0\mathrm{f}}+(1-\chi)\varkappa_{0\mathrm{s}}\nabla_y\Theta^i+\chi\mathbf{e}_i\big)=0.$$

Substituting expressions (4.13)–(4.16) into the macroscopic equations (4.5)–(4.8), we obtain

$$\mathbb{A}_1=\mu_0 m\sum_{i,j=1}^3 J^{ij}\otimes J^{ij}+\mu_0\mathbb{A}_0^\mathrm{f},\qquad \mathbb{A}_0^\mathrm{f}=\mu_0\sum_{i,j=1}^3\big\langle\mathbb{D}\big(y,\mathbf{W}_0^{ij}\big)\big\rangle_{Y_\mathrm{f}}\otimes J^{ij},\tag{4.17}$$

$$\mathbb{A}_2=\lambda_0(1-m)\sum_{i,j=1}^3 J^{ij}\otimes J^{ij}-\lambda_0\mathbb{A}_0^\mathrm{f}+\mu_0\mathbb{A}_1^\mathrm{f}(0),\qquad \mathbb{A}_3(t)=\mu_0\frac{d}{dt}\mathbb{A}_1^\mathrm{f}(t)-\lambda_0\mathbb{A}_1^\mathrm{f}(t),\tag{4.18}$$

$$\mathbb{A}_1^\mathrm{f}(t)=\sum_{i,j=1}^3\Big\{\mu_0\Big\langle\mathbb{D}\Big(y,\frac{\partial\mathbf{W}^{ij}}{\partial t}\Big)\Big\rangle_{Y_\mathrm{f}}+\lambda_0\big\langle\mathbb{D}\big(y,\mathbf{W}^{ij}\big)\big\rangle_{Y_\mathrm{s}}\Big\}\otimes J^{ij},\tag{4.19}$$

$$B(t)=\mu_0\Big\langle\mathbb{D}\Big(y,\frac{\partial\mathbf{W}^0}{\partial t}\Big)\Big\rangle_{Y_\mathrm{f}}+\lambda_0\big\langle\mathbb{D}\big(y,\mathbf{W}^0\big)\big\rangle_{Y_\mathrm{s}},\tag{4.20}$$

$$C_1(t)=-C_2(t)=\sum_{i,j=1}^3\big\langle\operatorname{div}_y\mathbf{W}^{ij}\big\rangle_{Y_\mathrm{f}}J^{ij},\qquad a(t)=\big\langle\operatorname{div}_y\mathbf{W}_1^\theta\big\rangle_{Y_\mathrm{s}},\tag{4.21}$$

$$a_1(t)=-a_2(t)=\big\langle\operatorname{div}_y\mathbf{W}^0\big\rangle_{Y_\mathrm{f}},\qquad a_1^\theta(t)=-a_2^\theta(t)=\big\langle\operatorname{div}_y\mathbf{W}^\theta\big\rangle_{Y_\mathrm{f}},\tag{4.22}$$

$$B^\theta(t)=\mu_0\Big\langle\mathbb{D}\Big(y,\frac{\partial\mathbf{W}^\theta}{\partial t}\Big)\Big\rangle_{Y_\mathrm{f}}+\lambda_0\big\langle\mathbb{D}\big(y,\mathbf{W}^\theta\big)\big\rangle_{Y_\mathrm{s}},\tag{4.23}$$

$$B_0^\theta=\widehat{\varkappa}_0\mathbb{I}+\sum_{i=1}^3\big\{\varkappa_{0\mathrm{f}}\langle\nabla\Theta^i\rangle_{Y_\mathrm{f}}+\varkappa_{0\mathrm{s}}\langle\nabla\Theta^i\rangle_{Y_\mathrm{s}}\big\}\otimes\mathbf{e}_i,\tag{4.24}$$

where $\widehat{\varkappa}_0=m\varkappa_{0\mathrm{f}}+(1-m)\varkappa_{0\mathrm{s}}$. □

**Lemma 4.4.** *The tensors* $\mathbb{A}_1$, $\mathbb{A}_2$, *and* $\mathbb{A}_3$, *the matrices* $B$, $B^\theta$, $B_0^\theta$, $C_1$, *and* $C_2$, *and the numbers* $a_1$, $a_2$, $a_1^\theta$, $a_2^\theta$, *and* $a$ *are well-defined infinitely differentiable functions of time.*

*If the porous space is connected, then the symmetric tensor* $\mathbb{A}_1$ *is strictly positive definite. Otherwise (if the pores are isolated),* $\mathbb{A}_1 = 0$ *and the symmetric tensor* $\mathbb{A}_2$ *is strictly positive definite. The symmetric matrix* $B_0^\theta$ *is strictly positive definite as well.*

The key points of the proof of the lemma, except the assertion concerning the matrix $B_0^\theta$, can be found in [8]. The properties of the matrix $B_0^\theta$ are well known (see [3, 15]).

## REFERENCES

1. A. M. Meirmanov and S. A. Sazhenkov, "Generalized Solutions to Linearized Equations of Thermoelastic Solid and Viscous Thermofluid," Electron. J. Diff. Eqns., 41 (2007).
2. R. Burridge and J. B. Keller, "Poroelasticity Equations Derived from Microstructure," J. Acoust. Soc. Am. **70** (4), 1140–1146 (1981).
3. E. Sanchez-Palencia, *Non-homogeneous Media and Vibration Theory* (Springer, New York, 1980; Mir, Moscow, 1984).
4. G. Nguetseng, "Asymptotic Analysis for a Stiff Variational Problem Arising in Mechanics," SIAM J. Math. Anal. **21**, 1394–1414 (1990).
5. R. P. Gilbert and A. Mikelić, "Homogenizing the Acoustic Properties of the Seabed. I," Nonlinear Anal., Theory Methods Appl. **40**, 185–212 (2000).
6. Th. Clopeau, J. L. Ferrín, R. P. Gilbert, and A. Mikelić, "Homogenizing the Acoustic Properties of the Seabed. II," Math. Comput. Modell. **33**, 821–841 (2001).
7. J. L. Ferrin and A. Mikelić, "Homogenizing the Acoustic Properties of a Porous Matrix Containing an Incompressible Inviscid Fluid," Math. Methods Appl. Sci. **26**, 831–859 (2003).
8. A. M. Meirmanov, "Nguetseng's Two-Scale Convergence Method for Filtration and Seismic Acoustic Problems in Elastic Porous Media," arXiv:math/0611330.
9. G. Nguetseng, "A General Convergence Result for a Functional Related to the Theory of Homogenization," SIAM J. Math. Anal. **20**, 608–623 (1989).
10. D. Lukkassen, G. Nguetseng, and P. Wall, "Two-Scale Convergence," Int. J. Pure Appl. Math. **2** (1), 35–86 (2002).
11. V. V. Zhikov, "Connectedness and Homogenization. Examples of Fractal Conductivity," Mat. Sb. **187** (8), 3–40 (1996) [Sb. Math. **187**, 1109–1147 (1996)].
12. V. V. Zhikov, "On an Extension of the Method of Two-Scale Convergence and Its Applications," Mat. Sb. **191** (7), 31–72 (2000) [Sb. Math. **191**, 973–1014 (2000)].
13. V. V. Zhikov, "Homogenization of Elasticity Problems on Singular Structures," Izv. Ross. Akad. Nauk, Ser. Mat. **66** (2), 81–148 (2002) [Izv. Math. **66**, 299–365 (2002)].
14. O. A. Ladyzhenskaya, *Mathematical Problems of Dynamics of Viscous Incompressible Flow*, 2nd ed. (Nauka, Moscow, 1970) [in Russian].
15. V. V. Zhikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals* (Nauka, Moscow, 1993; Springer, Berlin, 1994).