Homogenized models for filtration and for acoustic wave propagation in thermo-elastic porous media

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A system of differential equations describing the joint motion of thermo-elastic porous body and slightly compressible viscous thermofluid occupying pore space is considered. Although the problem is correct in an appropriate functional space, it is very hard to tackle due to the fact that its main differential equations involve non-smooth oscillatory coefficients, both big and small, under the differentiation operators. The rigorous justification under various conditions imposed on physical parameters is fulfilled for homogenization procedures as the dimensionless size of the pores tends to zero, while the porous body is geometrically periodic. As a result, we derive Biot's system of equations of thermo-poroelasticity, a similar system, consisting of anisotropic Lamé equations for a thermoelastic solid coupled with acoustic equations for a thermofluid, Darcy's system of filtration, or acoustic equations for a thermofluid, according to ratios between physical parameters. The proofs are based on Nguetseng's two-scale convergence method of homogenization in periodic structures.

1 Introduction

In this article a problem of modelling of small perturbations in a thermoelastic deformable solid, perforated by a system of channels (pores) filled with compressible viscous thermofluid, is considered. The solid component of such a medium is called *skeleton*, and the domain which is filled with a fluid is named a *pore space*. The exact mathematical model of such a medium consists of the classical equations of momentum, energy and mass balance, which are stated in Euler variables, of the equations determining stress fields and thermodynamics law in both solid and liquid phases, and of a relation determining the behaviour of the interface between liquid and solid components. This relation expresses the fact that the interface is a material surface, which amounts to the condition that it consists of the same material particles all the time. Clearly, the above-stated original model is a model with an unknown (free) boundary. The more precise formulation of the non-linear problem is not the focus of our present work, because the influence of the free boundary and the influence of convective terms in Navier-Stokes equations are negligible (the liquid velocity in pores is about six metres per year). Therefore we aim to study the problem linearized with respect to the stress tensor and displacements at the rest state. We suppose that the stress tensor is a linear function of the displacements and that in the liquid component the viscosity depends on temperature. In this model the characteristic function of the pore space $\tilde{\chi}$ is a known function for $t > 0$. It is assumed that this function coincides with the characteristic function of the pore space $\bar{\chi}$, given at the initial moment. The model obtained in this way when the viscosity doesn't depend on the temperature has been studied in [11].

In dimensionless variables (without primes),

$$\mathbf{x}' = L\mathbf{x}, \quad t' = \tau t, \quad \mathbf{w}' = L\mathbf{w}, \quad \theta' = \vartheta_* \theta,$$

the differential equations of the problem in the domain $\Omega \in \mathbb{R}^3$ for the dimensionless displacement vector $\mathbf{w}$ of the continuum medium and the dimensionless temperature θ have the form

$$\alpha_\tau \bar{\rho} \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname{div} \mathbb{P} + \bar{\rho} \mathbf{F}, \tag{1.1}$$

$$\bar{c}_p \frac{\partial \theta}{\partial t} = \operatorname{div}(\bar{\alpha}_\varkappa \nabla \theta) - \bar{\alpha}_\theta \frac{\partial}{\partial t}(\operatorname{div} \mathbf{w}), \tag{1.2}$$

$$\mathbb{P} = \bar{\chi} \mathbb{P}^f + (1 - \bar{\chi}) \mathbb{P}^s, \tag{1.3}$$

$$\mathbb{P}^f = \alpha_\mu \nu(\theta) \mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) - (p_f + \alpha_{\theta f} \theta) \mathbb{I}, \tag{1.4}$$

$$\mathbb{P}^s = \alpha_\lambda \mathbb{D}(x, \mathbf{w}) + (\alpha_\eta (\operatorname{div} \mathbf{w} - \alpha_{\theta s} \theta) \mathbb{I}, \tag{1.5}$$

$$p_f + \bar{\chi} \alpha_p \operatorname{div} \mathbf{w} = 0. \tag{1.6}$$

Here and subsequently we use notations

$$\mathbb{D}(x, \mathbf{u}) = (1/2) \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right), \quad \bar{\rho} = \bar{\chi} \rho_f + (1 - \bar{\chi}) \rho_s,$$

$$\bar{c}_p = \bar{\chi} c_{pf} + (1 - \bar{\chi}) c_{ps}, \quad \bar{\alpha}_\varkappa = \bar{\chi} \alpha_{\varkappa f} + (1 - \bar{\chi}) \alpha_{\varkappa s}, \quad \bar{\alpha}_\theta = \bar{\chi} \alpha_{\theta f} + (1 - \bar{\chi}) \alpha_{\theta s},$$

$\mathbb{I}$ is the unit tensor, the given function $\bar{\chi}(\mathbf{x})$ is a characteristic function of the pore space, the function $\mathbf{F}(\mathbf{x}, t)$ is a dimensionless vector of distributed mass forces, $\mathbb{P}^f$ is the liquid stress tensor, $\mathbb{P}^s$ is the stress tensor in the solid skeleton and p_f is the liquid pressure. The given function $\nu(\theta)$ is a dimensionless viscosity, such that $\nu(\theta_*) = 1$.

The differential equations (1.1)–(1.6) mean that the the displacement vector $\mathbf{w}$ and the temperature θ satisfy the non-isothermal Stokes equations in the pore space Ω_f ($\bar{\chi} = 1$) and the non-isothermal Lamé equations in the solid skeleton Ω_s ($\bar{\chi} = 0$). On the common 'solid skeleton–pore space' boundary Γ the displacement vector $\mathbf{w}$, the temperature θ and the liquid pressure p_f satisfy the usual continuity conditions

$$[\mathbf{w}](\mathbf{x}_0, t) = 0, \quad [\theta](\mathbf{x}_0, t) = 0, \ \ \mathbf{x}_0 \in \Gamma, \ t \geqslant 0, \tag{1.7}$$

the momentum conservation law and the heat conservation law in the form

$$[\mathbb{P} \cdot \mathbf{n}](\mathbf{x}_0, t) = 0, \quad [\bar{\alpha}_\varkappa \nabla \theta \cdot \mathbf{n}](\mathbf{x}_0, t) = 0, \ \ \mathbf{x}_0 \in \Gamma, \ t \geqslant 0, \tag{1.8}$$

where $\mathbf{n}(\mathbf{x}_0)$ is the unit normal to the boundary at the point $\mathbf{x}_0 \in \Gamma$ and

$$[\varphi](\mathbf{x}_0, t) = \varphi_{(s)}(\mathbf{x}_0, t) - \varphi_{(f)}(\mathbf{x}_0, t),$$

$$\varphi_{(s)}(\mathbf{x}_0, t) = \lim_{\substack{\mathbf{x} \to \mathbf{x}_0 \\ \mathbf{x} \in \Omega_s}} \varphi(\mathbf{x}, t), \quad \varphi_{(f)}(\mathbf{x}_0, t) = \lim_{\substack{\mathbf{x} \to \mathbf{x}_0 \\ \mathbf{x} \in \Omega_f}} \varphi(\mathbf{x}, t).$$

The problem is endowed with initial conditions

$$\mathbf{w}(\mathbf{x},0)=0, \quad \frac{\partial \mathbf{w}}{\partial t}(\mathbf{x},0)=v_0(\mathbf{x}), \quad \theta(\mathbf{x},0)=\theta_0(\mathbf{x}), \ \ \mathbf{x}\in\Omega \tag{1.9}$$

and homogeneous boundary conditions

$$\mathbf{w}(\mathbf{x},t)=0, \quad \theta(\mathbf{x},t)=0, \ \ \mathbf{x}\in S=\partial\Omega, \quad t\geqslant 0. \tag{1.10}$$

One may find the exact expression of the dimensionless constants α_i $(i=\tau,v,\ldots)$ in [11]. In particular,

$$\alpha_\tau=\frac{L}{g\tau^2}, \ \ \alpha_\mu=\frac{2v_*}{\tau g\rho_0 L}, \ \ \alpha_\lambda=\frac{2\lambda}{g\rho_0 L}, \ \ \alpha_p=\frac{c^2\rho_f}{Lg}, \ \ \alpha_\eta=\frac{\eta}{Lg\rho_0},$$

where v_* is the viscosity of fluid at the temperature $\theta=\theta_*$, λ and η are the elastic Lamé constants, $\varkappa_f$ and $\varkappa_s$ are heat conductivities in the liquid and in the solid components respectively, c is a speed of sound in fluid, L is a characteristic size of the domain in consideration, τ is a characteristic time of the process, ρ_f and ρ_s are the mean dimensionless densities of the liquid and rigid phases, respectively, scaled with the mean density of water, and g is the value of acceleration of gravity.

From a purely mathematical point of view, the corresponding initial-boundary value problem (1.1)–(1.10) is well posed in the sense that it has a unique solution belonging to a suitable functional space on any finite time interval (see [11]). However, regarding possible applications, for example, for developing numerical codes, this model is inappropriate because of its complexity even if a modern supercomputer is available. The differential equations of the model involve rapidly oscillating non-smooth coefficients, which have the form of linear combinations of the function $\bar{\chi}$. These coefficients undergo differentiation with respect to x and besides may be very big or very small quantities as compared to the main small parameter ε. In the model under consideration we define the dimensionless size of the pores ε as the characteristic size of pores l divided by the characteristic size L of the entire porous body:

$$\varepsilon=\frac{l}{L}.$$

Therefore the question of finding an effective approximate model is vital. Since the model involves the small parameter ε, the most natural approach to this problem is to derive models that would describe limiting regimes arising as ε tends to zero. Such approximations significantly simplify the original problem and at the same time preserve all of its main features. But even this approach is too hard to work out, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, the most appropriate is to simplify the problem postulating that the porous structure is periodic.

We assume the following constraints.

Assumption 1 The domain $\Omega=(0,1)^3$ is a periodic repetition of an elementary cell $Y^\varepsilon=\varepsilon Y$, where $Y=(0,1)^3$ and the quantity $1/\varepsilon$ is an integer, so that Ω always contains an integer number of elementary cells Y^ε. Let Y_s be a 'solid part' of Y, and the 'liquid part' Y_f be its open complement. We write $\gamma=\partial Y_f\cap\partial Y_s$ and take γ as a

Lipschitz-continuous surface. A pore space Ω_f^ε is a periodic repetition of the elementary cell εY_f, and a solid skeleton Ω_s^ε is a periodic repetition of the elementary cell εY_s. A Lipschitz-continuous boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition in Ω of the boundary $\varepsilon\gamma$. The solid skeleton Ω_s^ε is a connected domain and an intersection Ω_s^ε with any plane $\{x_i = constant,\ 0 < x_i < 1,\ i = 1,2,3\}$ is an open (in plane topology) set.

In these assumptions

$$\bar{\chi}(\mathbf{x}) = \chi^\varepsilon(\mathbf{x}) = \chi(\mathbf{x}/\varepsilon),$$

$$\bar{\rho} = \rho^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\rho_f + (1 - \chi^\varepsilon(\mathbf{x}))\rho_s,$$

$$\bar{c}_p = c_p^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})c_{pf} + (1 - \chi^\varepsilon(\mathbf{x}))c_{ps},$$

$$\bar{\alpha}_\varkappa = \alpha_\varkappa^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\alpha_{\varkappa f} + (1 - \chi^\varepsilon(\mathbf{x}))\alpha_{\varkappa s},$$

$$\bar{\alpha}_\theta = \alpha_\theta^\varepsilon(\mathbf{x}) = \chi^\varepsilon(\mathbf{x})\alpha_{\theta f} + (1 - \chi^\varepsilon(\mathbf{x}))\alpha_{\theta s},$$

where $\chi(\mathbf{y})$ is the characteristic function of Y_f in Y.

We say that a *pore space is disconnected (isolated pores)* if

$$\gamma \cap \partial Y = \emptyset.$$

Suppose that all the dimensionless parameters depend on the small parameter ε and there exist limits (finite or infinite)

$$\lim_{\varepsilon\searrow 0} \alpha_\tau(\varepsilon) = \tau_0,\ \ \lim_{\varepsilon\searrow 0} \alpha_\mu(\varepsilon) = \mu_0,\ \ \lim_{\varepsilon\searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0,\ \ \lim_{\varepsilon\searrow 0} \frac{\alpha_\mu}{\varepsilon^2} = \mu_1.$$

The first research with the aim of finding limiting regimes in the case when the skeleton was assumed to be an absolutely rigid isothermal body was carried out by E. Sanchez-Palencia and L. Tartar (see [14]). E. Sanchez-Palencia [14, Sec. 7.2] formally obtained Darcy's law of filtration using the method of two-scale asymptotic expansions, and L. Tartar [14, Appendix] mathematically rigorously justified the homogenization procedure. Using the same method of two-scale expansions, J. Keller and R. Burridge [2] derived formally the system of Biot equations [3] from the isothermal model (1.1)–(1.10) ($\alpha_\theta^\varepsilon = 0$) in the case when $0 < \mu_1 < \infty$, and the rest of the coefficients were fixed independent of ε. It is well known that the various modifications of Biot's model are the bases of seismic acoustics problems to date. This fact emphasizes the importance of a comprehensive study of the model (1.1)–(1.10) one more time. J. Keller and R. Burridge [2] also considered the same problem under the assumption that all the physical parameters were fixed independent of ε and formally derived, as a result, a system of equations of visco-elasticity.

Under the same assumptions as in the article [2], the rigorous justification of Biot's model was given by G. Nguetseng [13] and later by Th. Clopeau *et al.* in [4]. Also R. Gilbert and Mikelić [5] have rigorously derived a system of equations of viscoelasticity, when all the physical parameters were fixed independent of ε. The most general case of the isothermal model has been studied in [8]. In these papers Nguetseng's two-scale convergence method [7, 12] was the main method of investigation.

In the present publication we investigate all possible limiting regimes in the problem (1.1)–(1.10) by means of this method. This method, in a rather simple form, reveals the structure of the weak limit of a sequence $\{z^\varepsilon\}$ as $\varepsilon \searrow 0$, where $z^\varepsilon = u^\varepsilon v^\varepsilon$ and sequences $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ converge as $\varepsilon \searrow 0$ merely weakly, but at the same time the function u^ε has the special structure $u^\varepsilon(\mathbf{x}) = u(\mathbf{x}/\varepsilon)$ with $u(\mathbf{y})$ being periodic in $\mathbf{y}$.

Moreover, this method allows us to establish asymptotic expansions of a solution of the problem (1.1)–(1.10) in the form

$$\mathbf{w}^\varepsilon(\mathbf{x},t) = \varepsilon^\beta \left(\mathbf{w}_0(\mathbf{x},t) + \varepsilon\mathbf{w}_1\left(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}\right) + o(\varepsilon)\right), \tag{1.11}$$

where $\mathbf{w}_0(\mathbf{x},t)$ is a solution of the homogenized (limiting) problem, $\mathbf{w}_1(\mathbf{x},t,\mathbf{y})$ is a solution of some initial-boundary value problem posed on the generic periodic cell of the pore space and the exponent β is defined by dimensionless parameters of the model. In some situations expansion (1.11) has a more complicated form like

$$\mathbf{w}^\varepsilon(\mathbf{x},t) = \varepsilon^{\beta_f} \left(\mathbf{w}_0^f(\mathbf{x},t) + \varepsilon\mathbf{w}_1^f\left(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}\right) + o(\varepsilon)\right),$$

in the liquid component and

$$\mathbf{w}^\varepsilon(\mathbf{x},t) = \varepsilon^{\beta_s} \left(\mathbf{w}_0^s(\mathbf{x},t) + \varepsilon\mathbf{w}_1^s\left(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}\right) + o(\varepsilon)\right),$$

in the rigid component.

We restrict our consideration to the cases with $\tau_0 < \infty$ and one of the following occurs:

$$\textbf{(I)} \qquad \mu_0 = 0, \qquad 0 < \lambda_0 < \infty;$$
$$\textbf{(II)} \qquad 0 \leqslant \mu_1 < \infty, \qquad \lambda_0 = \infty.$$

If $\tau_0 = \infty$, re-normalizing the displacement vector and temperature by setting

$$\mathbf{w} \to \alpha_\tau \mathbf{w}, \quad \theta \to \alpha_\tau \theta,$$

we reduce the problem to one of cases (I) and (II).

Note that anisothermic case (II) for the inhomogeneous boundary conditions has been considered in [9] and that anisothermic cases (I) and (II) for incompressible media have been considered in [10].

2 Formulation of the main results

There are various ways, equivalent in the sense of distributions, of representing equations (1.1)–(1.2) in each domain Ω_f^ε and Ω_s^ε and boundary conditions (1.7)–(1.8) on the common boundary Γ^ε between pore space Ω_f^ε and solid skeleton Ω_s^ε. In what follows, it is convenient to write them in the form of the integral equalities.

Definition 1 We say that $\left(\mathbf{w}^\varepsilon, \theta^\varepsilon, p_f^\varepsilon, p_s^\varepsilon\right)$ is a generalized solution of the problem (1.1)–(1.10), if the following are satisfied:

(i) $\nabla\mathbf{w}^{\varepsilon}$, $\nabla\theta^{\varepsilon}$, p_f^{ε}, $p_s^{\varepsilon} \in L^2(\Omega_T)$ in the domain $\Omega_T = \Omega \times (0, T)$;

(ii) the boundary condition (1.10) on the outer boundary S in the trace sense;

(iii) the equations

$$\frac{1}{\alpha_p} p_f^{\varepsilon} = -\chi^{\varepsilon} \mathrm{div}\mathbf{w}^{\varepsilon}, \tag{2.1}$$

$$\frac{1}{\alpha_\eta} p_s^{\varepsilon} = -(1-\chi^{\varepsilon})\,\mathrm{div}\mathbf{w}^{\varepsilon} \tag{2.2}$$

a.e. in Ω_T;

(iv) the integral identity

$$\begin{aligned}
&\int_{\Omega_T} \Big(\rho^{\varepsilon}\alpha_\tau \mathbf{w}^{\varepsilon} \cdot \frac{\partial^2\varphi}{\partial t^2} - \chi^{\varepsilon}\alpha_\mu \nu(\theta)\mathbb{D}(x, \mathbf{w}^{\varepsilon}) : \mathbb{D}\Big(x, \frac{\partial\varphi}{\partial t}\Big) - \rho^{\varepsilon}\mathbf{F}\cdot\varphi \\
&\qquad + \{(1-\chi^{\varepsilon})\alpha_\lambda \mathbb{D}(x, \mathbf{w}^{\varepsilon}) - (p_f^{\varepsilon} + p_s^{\varepsilon} + \alpha_\theta^{\varepsilon}\theta^{\varepsilon})\mathbb{I}\} : \mathbb{D}(x, \varphi)\Big)\, dx\, dt \\
&= \int_{\Omega} \rho^{\varepsilon}\alpha_\tau v_0^{\varepsilon}(\mathbf{x}) \cdot \varphi(\mathbf{x}, 0)\, dx
\end{aligned} \tag{2.3}$$

for all smooth vector-functions $\varphi = \varphi(\mathbf{x}, t)$ such that

$$\varphi(\mathbf{x}, t) = 0, \;\; \mathbf{x} \in S, \;\; t > 0; \quad \varphi(\mathbf{x}, T) = \frac{\partial\varphi}{\partial t}(\mathbf{x}, T) = 0, \;\; \mathbf{x} \in \Omega;$$

and

(v) the integral identity

$$\int_{\Omega_T} \Big(\big(c_p^{\varepsilon}\theta^{\varepsilon} + \alpha_\theta^{\varepsilon}\,\mathrm{div}\,\mathbf{w}^{\varepsilon}\big)\frac{\partial\xi}{\partial t} - \alpha_\varkappa^{\varepsilon}\nabla\theta^{\varepsilon}\cdot\nabla\xi\Big)\, dx\, dt = -\int_{\Omega} c_p^{\varepsilon}\theta_0^{\varepsilon}(\mathbf{x})\xi(\mathbf{x}, 0)\, dx \tag{2.4}$$

for all smooth functions $\xi = \xi(\mathbf{x}, t)$ such that

$$\xi(\mathbf{x}, t) = 0, \;\; \mathbf{x} \in S, \;\; t > 0; \quad \xi(\mathbf{x}, T) = 0, \;\; \mathbf{x} \in \Omega.$$

In this definition we changed the form of representation of the stress tensor $\mathbb{P}$ in the integral identity (2.3) by introducing a new unknown function p_s^{ε}, which is something like a pressure. In what follows we call this function p_s^{ε} a solid pressure and regard equations (2.1) and (2.2) as continuity equations. This special choice of continuity equations simplifies the use of the homogenization procedure.

In (2.3) $A : B$ denotes the convolution (or, equivalently, the inner tensor product) of two second-rank tensors along both the indexes, i.e., $A : B = \mathrm{tr}\,(B^* \circ A) = \sum_{i,j=1}^{3} A_{ij}B_{ji}$.

Suppose additionally that there exist limits (finite or infinite)

$$\lim_{\varepsilon\searrow 0}\alpha_p(\varepsilon) = p_*, \;\; \lim_{\varepsilon\searrow 0}\alpha_\eta(\varepsilon) = \eta_0, \quad \lim_{\varepsilon\searrow 0}\alpha_{\varkappa i}(\varepsilon) = \varkappa_{0i}, \quad \lim_{\varepsilon\searrow 0}\alpha_{\theta i}(\varepsilon) = \beta_{0i},$$

where $i = f, s$.

We assume also

Assumption 2

(1) The dimensionless parameters satisfy the restrictions

$$\varkappa_{0i},\ \beta_{0i},\ p_*,\ \eta_0,\ \tau_0,\ \mu_0 < \infty, \quad 0 < \varkappa_{0i},\ \tau_0 + \mu_1,\ \lambda_0,\ p_*,\ \eta_0,$$

where $i = f, s$.

(2) The functions $\mathbf{F}$ and $\partial \mathbf{F}/\partial t$ are bounded in $L^2(\Omega_T)$ and the sequences $\{\mathbf{v}_0^\varepsilon\}$ and $\{\theta_0^\varepsilon\}$ converge strongly in $L^2(\Omega)$ to the functions $\mathbf{v}_0$ and θ_0, respectively.

(3) The smooth function $\nu(\theta)$ is strictly positive and bounded for all θ:

$$0 < \nu_0 < \nu(\theta) < \nu_0^{-1}, \quad \nu \in C^2(-\infty, \infty).$$

In what follows all parameters may take all permitted values. For example, if $\tau_0 = 0$ or $\beta_{0s} = 0$, then all terms in the homogenized equations containing these parameters disappear.

The following theorems (Theorems 1–3) are the main results of the paper.

Theorem 1 *For all $\varepsilon > 0$ on the arbitrary time interval $[0, T]$ there exists a generalized solution of the problem (1.1)–(1.10) and the following assertions hold:*

$$\max_{0\leqslant t\leqslant T} \| |\mathbf{w}^\varepsilon(t)| + \sqrt{\alpha_\mu}\chi^\varepsilon |\nabla \mathbf{w}^\varepsilon(t)| + \sqrt{\alpha_\lambda}(1-\chi^\varepsilon)|\nabla \mathbf{w}^\varepsilon(t)| \|_{2,\Omega} \leqslant C, \tag{2.5}$$

$$\max_{0\leqslant t\leqslant T} \|\theta^\varepsilon(t)\|_{2,\Omega} + \|\nabla\theta^\varepsilon\|_{2,\Omega_T} \leqslant C, \tag{2.6}$$

$$\|p_f^\varepsilon\|_{2,\Omega_T} + \|p_s^\varepsilon\|_{2,\Omega_T} \leqslant C, \tag{2.7}$$

where C is a constant independent of the small parameter ε.

Theorem 2 *Let $\nu(\theta) \equiv 1$, $\mu_0 = 0$ and $\lambda_0 < \infty$. Then functions $\mathbf{w}^\varepsilon$ admit an extension $\mathbf{u}^\varepsilon$ from $\Omega_s^\varepsilon \times (0, T)$ into Ω_T and there exist a subsequence of small parameters $\{\varepsilon > 0\}$ and functions $\mathbf{u}$, $\mathbf{w}$, p_f, p_s and θ, such that the sequence $\{\mathbf{u}^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2((0, T); W_2^1(\Omega))$ to the function $\mathbf{u} \in \overset{\circ}{W}{}_2^1(\Omega_T)$. At the same time sequences $\{\mathbf{w}^\varepsilon\}$, $\{p_f^\varepsilon\}$ and $\{p_s^\varepsilon\}$ converge weakly in $L^2(\Omega_T)$ to $\mathbf{w}$, p_f and p_s, respectively and sequence $\{\theta^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2(((0, T); \overset{\circ}{W}{}_2^1(\Omega)))$ to the function θ.*

The following assertions for these limiting functions hold:

***(I)** If $\mu_1 = \infty$ or the pore space is disconnected, then $\mathbf{w} = \mathbf{u}$ and the strong and weak limits $\mathbf{u}$, θ, p_f and p_s satisfy in Ω_T the initial-boundary value problem*

$$\tau_0 \hat{\rho} \frac{\partial^2 \mathbf{u}}{\partial t^2} + \nabla(p_f + p_s + \hat{\beta}_0 \theta) - \hat{\rho}\mathbf{F} = \operatorname{div}\big(\lambda_0 \mathbb{A}_0^s : \mathbb{D}(x, \mathbf{u}) + \mathbb{B}_0^s \operatorname{div} \mathbf{u} + \mathbb{B}_1^s(\beta\theta + p_f)\big), \tag{2.8}$$

$$\hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*}\frac{\partial p_f}{\partial t} - \frac{\beta_{0s}}{\eta_0}\frac{\partial p_s}{\partial t} = \operatorname{div}(\mathbb{B}^\theta \cdot \nabla\theta), \tag{2.9}$$

$$\frac{1}{\eta_0} p_s + \mathbb{C}_0^s : \mathbb{D}(x, \mathbf{u}) + a_0^s \operatorname{div} \mathbf{u} + a_1^s(\beta\theta + p_f) = 0, \tag{2.10}$$

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \operatorname{div}\mathbf{u} = 0. \tag{2.11}$$

Here

$$\hat{\rho} = m\rho_f + (1-m)\rho_s, \quad \hat{c_p} = mc_{pf} + (1-m)c_{ps},$$

$$\hat{\beta}_0 = m\beta_{0f} + (1-m)\beta_{0s}, \quad \beta = m(\beta_{0f} - \beta_{0s}), \quad m = \int_Y \chi(\mathbf{y})\,dy,$$

and the symmetric strictly positively defined constant fourth-rank tensor $\mathbb{A}_0^s$, *constant matrices* $\mathbb{C}_0^s, \mathbb{B}_0^s$ *and* $\mathbb{B}_1^s$, *strictly positively defined constant matrix* $\mathbb{B}^\theta$ *and constants* a_0^s *and* a_1^s *are defined by (5.31)–(5.33) and (5.49).*

The differential equations (2.8)–(2.11) are endowed with initial conditions at $t = 0$ *and* $\mathbf{x} \in \Omega$

$$\theta = \theta_0, \quad \tau_0\mathbf{u} = \tau_0\left(\frac{\partial \mathbf{u}}{\partial t} - \mathbf{v}_0\right) = 0, \tag{2.12}$$

and homogeneous boundary conditions

$$\theta(\mathbf{x}, t) = 0, \quad \mathbf{u}(\mathbf{x}, t) = 0, \ \ \mathbf{x} \in S, \quad t > 0. \tag{2.13}$$

(II) *If* $\mu_1 < \infty$, *then the strong and weak limits* $\mathbf{u}$, θ, $\mathbf{w}^f$, p_f *and* p_s *of the sequences* $\{\mathbf{u}^\varepsilon\}$, $\{\theta^\varepsilon\}$, $\{\chi^\varepsilon\mathbf{w}^\varepsilon\}$, $\{p_f^\varepsilon\}$ *and* $\{p_s^\varepsilon\}$ *satisfy the initial-boundary value problem in* Ω_T, *consisting of the heat equation (2.9), the balance of momentum equation*

$$\tau_0\left(\rho_f\frac{\partial \mathbf{v}}{\partial t} + \rho_s(1-m)\frac{\partial^2 \mathbf{u}}{\partial t^2}\right) + \nabla(p_f + p_s + \hat{\beta}_0\theta) - \hat{\rho}\mathbf{F}$$
$$= \operatorname{div}\big(\lambda_0\mathbb{A}_0^s : \mathbb{D}(x, \mathbf{u}) + \mathbb{B}_0^s \operatorname{div}\mathbf{u} + \mathbb{B}_1^s(\beta\theta + p_f)\big), \tag{2.14}$$

where $\mathbf{v} = \partial\mathbf{w}^f/\partial t$ *and* $\mathbb{A}_0^s$, $\mathbb{B}_0^s$ *and* $\mathbb{B}_1^s$ *are the same as in (2.8), the continuity equation (2.10) for the solid component and the continuity equation*

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t} + \frac{1}{\eta_0}\frac{\partial p_s}{\partial t} + \operatorname{div}\mathbf{v} = (m-1)\operatorname{div}\frac{\partial \mathbf{u}}{\partial t}, \tag{2.15}$$

for the liquid component, and either the relation

$$\mathbf{v} = m\frac{\partial \mathbf{u}}{\partial t} + \int_0^t \mathbb{B}_1(t-\tau)\cdot\mathbf{h}(\mathbf{x}, \tau)\,d\tau, \tag{2.16}$$

where

$$\mathbf{h} = -\nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f\mathbf{F} - \tau_0\rho_f\frac{\partial^2 \mathbf{u}}{\partial t^2},$$

in the case of $\tau_0 > 0$ *and* $\mu_1 > 0$; *or Darcy's law in the form*

$$\mathbf{v} = m\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\mu_1}\mathbb{B}_2\cdot\left(-\nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f\mathbf{F}\right) \tag{2.17}$$

in the case of $\tau_0 = 0$; *or, finally, the balance of momentum equation in the form*

$$\tau_0\rho_f\frac{\partial \mathbf{v}}{\partial t} = \tau_0\rho_f\mathbb{B}_3\cdot\frac{\partial^2 \mathbf{u}}{\partial t^2} + (m\mathbb{I} - \mathbb{B}_3)\cdot\left(-\nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f\mathbf{F}\right) \tag{2.18}$$

in the case of $\mu_1 = 0$ *for the liquid component.*

The problem is supplemented by boundary and initial conditions (2.12)–(2.13) for displacement $\mathbf{u}$ *of the rigid component and for temperature* θ *and by the initial and boundary conditions*

$$\tau_0(\mathbf{v}(\mathbf{x},0) - m\mathbf{v}_0(\mathbf{x})) = 0, \quad \mathbf{x} \in \Omega, \tag{2.19}$$

$$\mathbf{v}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) = 0, \quad \mathbf{x} \in S, \quad t > 0 \tag{2.20}$$

for the velocity $\mathbf{v}$ *of the liquid component. In (2.16)–(2.20)* $\mathbf{n}(\mathbf{x})$ *is the unit normal vector to* S *at a point* $\mathbf{x} \in S$, *and matrix* $\mathbb{B}_1(t)$, *symmetric strictly positively defined matrices* $\mathbb{B}_2$ *and* $(m\mathbb{I} - \mathbb{B}_3)$ *are defined below by formulas (5.42), (5.44) and (5.46).*

Theorem 3 *Let* $\lambda_0 = \infty$ *and* $\mu_1 < \infty$. *Then the functions* $\mathbf{w}^\varepsilon$ *admit an extension* $\mathbf{u}^\varepsilon$ *from* $\Omega_s^\varepsilon \times (0,T)$ *into* Ω_T *and there exist a subsequence of small parameters* $\{\varepsilon > 0\}$ *and functions* $\mathbf{w}^f$, p_f *and* θ, *such that the sequence* $\{\mathbf{u}^\varepsilon\}$ *converges strongly in* $L^2((0,T); W_2^1(\Omega))$ *to zero. At the same time sequences* $\{\chi^\varepsilon \mathbf{w}^\varepsilon\}$ *and* $\{p_f^\varepsilon\}$ *converge weakly in* $L^2(\Omega_T)$ *to* $\mathbf{w}^f$ *and* p_f *respectively and sequence* $\{\theta^\varepsilon\}$ *converges strongly in* $L^2(\Omega_T)$ *and weakly in* $L^2((0,T); \overset{\circ}{W}{}_2^1(\Omega))$ *to the function* θ.

(1) If $\tau_0 > 0$ *and* $\mu_1 > 0$, *then the functions* $\mathbf{v} = \partial \mathbf{w}^f/\partial t$, θ *and* p_f *solve the problem* (F_1), *which consists of the continuity equation*

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t} + \operatorname{div} \mathbf{v} = 0, \tag{2.21}$$

the heat equation

$$\hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*}\frac{\partial p_f}{\partial t} = \operatorname{div}(\mathbb{B}^\theta \cdot \nabla \theta), \tag{2.22}$$

and the relation

$$v(\mathbf{x},t) = m\mathbf{v}_0(\mathbf{x}) + \int_0^\tau \mathbb{B}_1(\tau - \xi) \cdot \tilde{\mathbf{z}}(\mathbf{x},\xi)\, d\xi, \tag{2.23}$$

where

$$\tilde{\mathbf{z}}\big(\mathbf{x}, \tau(\mathbf{x},t)\big) = \mathbf{z}(\mathbf{x},t) \equiv -\frac{1}{\mu_1 \nu(\theta)}\left(\nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f \mathbf{F}\right),$$

and the function $\tau(\mathbf{x},t)$ *is defined by the Cauchy problem*

$$\frac{\partial \tau}{\partial t} = \nu\big(\theta(\mathbf{x},t)\big), \quad \tau(\mathbf{x},0) = 0. \tag{2.24}$$

(2) If $\tau_0 = 0$ *and* $\mu_1 > 0$, *then the functions* $\mathbf{v}$, θ *and* p_f *solve the problem* (F_2), *which consists of equations (2.21) and (2.22) and Darcy's law in the form*

$$\mathbf{v} = \frac{1}{\mu_1 \nu(\theta)}\mathbb{B}_2 \cdot \left(-\nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f \mathbf{F}\right). \tag{2.25}$$

Finally,

(3) if $\tau_0 > 0$ *and* $\mu_1 = 0$, *then the functions* $\mathbf{w}^f$, θ *and* p_f *solve the problem* (F_3), *which consists of equations (2.21) and (2.22) and the balance of momentum equation (2.18), where* $\mathbf{u} = 0$.

In (2.22)–(2.25), matrices $\mathbb{B}^\theta$, $\mathbb{B}_1(t)$ *and* $\mathbb{B}_2$ *are the same as in Theorem 2.*

Problems F_1–F_3 are endowed with initial and boundary conditions (2.19) and (2.20) for the velocity in the liquid component and initial and boundary conditions (2.12) and (2.13) for the temperature.

3 Preliminaries

3.1 Two-scale convergence

Justification of Theorems 2–3 relies on systematic use of the method of two-scale convergence, which had been proposed by G. Nguetseng [12] and has been applied recently to a wide range of homogenization problems (see, for example, the survey [7]).

Definition 2 *A sequence $\{w^{\varepsilon}\} \subset L^2(\Omega_T)$ is said to be two-scale convergent to a 1-periodic in $\mathbf{y}$ function $W(\mathbf{x},t,\mathbf{y}) \in L^2(\Omega_T \times Y)$, if and only if for any 1-periodic in $\mathbf{y}$ function $\sigma = \sigma(\mathbf{x},t,\mathbf{y}) \in L^2(\Omega_T \times Y)$,*

$$\int_{\Omega_T} w^{\varepsilon}(\mathbf{x},t)\,\sigma\left(\mathbf{x},t,\frac{\mathbf{x}}{\varepsilon}\right) dx\,dt \to \int_{\Omega_T}\int_Y W(\mathbf{x},t,\mathbf{y})\,\sigma(\mathbf{x},t,\mathbf{y})\,dy\,dx\,dt \qquad (3.1)$$

as $\varepsilon \to 0$.

Existence and main properties of weakly convergent sequences are established by the following fundamental theorem [7, 12]:

Theorem 4 *(Nguetseng's theorem)*
(1) Any sequence bounded in $L^2(\Omega_T)$ contains a subsequence, two-scale convergent to some 1-periodic in $\mathbf{y}$ function $W(\mathbf{x},t,\mathbf{y}) \in L^2(\Omega_T \times Y)$.

(2) Let the sequences $\{w^{\varepsilon}\}$ and $\{\varepsilon\nabla_x w^{\varepsilon}\}$ be uniformly bounded in $L^2(\Omega_T)$. Then there exist a 1-periodic in $\mathbf{y}$ function $W = W(\mathbf{x},t,\mathbf{y})$ and a subsequence $\{w^{\varepsilon}\}$ such that $W, \nabla_y W \in L^2(\Omega_T \times Y)$, and the subsequences $\{w^{\varepsilon}\}$ and $\{\varepsilon\nabla_x w^{\varepsilon}\}$ two-scale converge to W and $\nabla_y W$, respectively.

(3) Let the sequences $\{w^{\varepsilon}\}$ and $\{\nabla_x w^{\varepsilon}\}$ be bounded in $L^2(Q)$. Then there exist functions $w \in L^2(\Omega_T)$ and $W \in L^2(\Omega_T \times Y)$ and a subsequence from $\{\nabla_x w^{\varepsilon}\}$ such that the function W is 1-periodic in $\mathbf{y}$, $\nabla_x w \in L^2(\Omega_T)$, $\nabla_y W \in L^2(\Omega_T \times Y)$, and the subsequence $\{\nabla_x w^{\varepsilon}\}$ two-scale converges to the function $(\nabla_x w(\mathbf{x},t) + \nabla_y W(\mathbf{x},t,\mathbf{y}))$.

Corollary 1 *Let $\sigma \in L^2(Y)$ and $\sigma^{\varepsilon}(\mathbf{x}) = \sigma(\mathbf{x}/\varepsilon)$. Assume that a sequence $\{w^{\varepsilon}\} \subset L^2(\Omega_T)$ two-scale converges to $W \in L^2(\Omega_T \times Y)$. Then the sequence $\{\sigma^{\varepsilon} w^{\varepsilon}\}$ two-scale converges to the function σW.*

3.2 An extension lemma

The typical difficulty in homogenization problems, like problem (1.1)–(1.10), while passing to a limit as $\varepsilon \searrow 0$ arises because the bounds on the gradient of displacement $\nabla \mathbf{w}^{\varepsilon}$ may be distinct for the liquid and rigid components. The classical approach in overcoming this difficulty consists of constructing an extension to the whole of Ω of the displacement field defined merely on Ω_s or Ω_f. The following lemma is valid due to the well-known results from [1, 6]. We formulate it in an appropriate form for us:

Lemma 3.1 *Suppose that assumption 1 on the geometry of the periodic structure holds,* $\mathbf{w}^\varepsilon \in W_2^1(\Omega_s^\varepsilon)$ *and* $\mathbf{w}^\varepsilon = 0$ *on* $S_s^\varepsilon = \partial\Omega_f^\varepsilon \cap \partial\Omega$ *in the sense of traces. Then there exists a function* $\mathbf{u}^\varepsilon \in W_2^1(\Omega)$ *such that its restriction to the sub-domain* Ω_s^ε *coincides with* $\mathbf{w}^\varepsilon$*, i.e.,*

$$(1 - \chi^\varepsilon(\mathbf{x}))(\mathbf{u}^\varepsilon(\mathbf{x}) - \mathbf{w}^\varepsilon(\mathbf{x})) = 0, \quad \mathbf{x} \in \Omega, \tag{3.2}$$

and, moreover, the estimates

$$\|\mathbf{u}^\varepsilon\|_{2,\Omega} \leqslant C\|\mathbf{w}^\varepsilon\|_{2,\Omega_s^\varepsilon}, \qquad \|\nabla\mathbf{u}^\varepsilon\|_{2,\Omega} \leqslant C\|\nabla\mathbf{w}^\varepsilon\|_{2,\Omega_s^\varepsilon} \tag{3.3}$$

hold, with the constant C *depending only on the geometry* Y *and not on* ε.

3.3 The Friedrichs–Poincaré inequality in periodic structure

The following lemma was proved by L. Tartar in [14, Appendix]. It specifies the Friedrichs–Poincaré inequality for ε-periodic structure. We formulate this lemma for our particular case just to estimate functions in the ε–layer Q^ε of the boundary S. This domain Q^ε consists of all elementary cells εY touching the boundary $\partial\Omega$. We consider a special class of functions $\mathbf{u}^\varepsilon$, which are extensions, from subdomain Ω_s^ε onto whole domain Ω, of functions $\mathbf{w}^\varepsilon \in W_2^1(\Omega_s^\varepsilon)$ vanishing on the part $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ of the boundary $S = \partial\Omega$ (see Lemma 3.1). Due to the supposition on the structure of the pore space, the intersection of the boundary of the 'solid part' Y_s with sides of the boundary ∂Y is a set with non-empty interior and strictly positive measure. Therefore on each side of the boundary S the function $\mathbf{u}^\varepsilon$ is equal to zero on some set with non-empty interior, periodic structure and strictly positive measure, independent of ε.

Lemma 3.2 *Suppose that the assumptions on the geometry of* Ω_f^ε *hold. Then for any function* $\mathbf{u}^\varepsilon \in W_2^1(\Omega)$ *such that* $\mathbf{u}^\varepsilon = 0$ *on the part* $S_s^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega$ *of the boundary* S *and for any function* $\mathbf{v}^\varepsilon \in \overset{\circ}{W}{}_2^1(\Omega_f^\varepsilon)$ *the inequalities*

$$\int_{Q^\varepsilon} |\mathbf{u}^\varepsilon|^2\, dx \leqslant C\varepsilon^2 \int_{Q^\varepsilon} |\nabla\mathbf{u}^\varepsilon|^2\, dx \tag{3.4}$$

and

$$\int_{\Omega_f^\varepsilon} |\mathbf{v}^\varepsilon|^2\, dx \leqslant C\varepsilon^2 \int_{\Omega_f^\varepsilon} |\nabla\mathbf{v}^\varepsilon|^2\, dx \tag{3.5}$$

hold with some constant C *independent of the small parameter* ε.

3.4 Some notation

We define

(1)

$$\langle \Phi \rangle_Y = \int_Y \Phi\, dy, \quad \langle \Phi \rangle_{Y_f} = \int_Y \chi \Phi\, dy, \quad \langle \Phi \rangle_{Y_s} = \int_Y (1-\chi)\Phi\, dy,$$

$$\langle \varphi \rangle_\Omega = \int_\Omega \varphi\, dx, \quad \langle \varphi \rangle_{\Omega_T} = \int_{\Omega_T} \varphi\, dx\, dt.$$

(2) If **a** and **b** are two vectors then the matrix $\mathbf{a} \otimes \mathbf{b}$ is defined by the formula

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

for any vector **c**.
(3) If $\mathbb{B}$ and $\mathbb{C}$ are two matrices, then $\mathbb{B} \otimes \mathbb{C}$ is a fourth-rank tensor such that its convolution with any matrix $\mathbb{A}$ is defined by the formula

$$(\mathbb{B} \otimes \mathbb{C}) : \mathbb{A} = \mathbb{B}(\mathbb{C} : \mathbb{A}).$$

(4) By $\mathbb{I}^{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$ we denote the 3×3 matrix with just one non-vanishing entry, which is equal to 1 and stands in the i-th row and the j-th column.
(5) We also introduce the matrices

$$\mathbb{J}^{ij} = \frac{1}{2}(\mathbb{I}^{ij} + \mathbb{I}^{ji}) = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \quad \mathbb{J} = \sum_{i,j=1}^{3} \mathbb{J}^{ij} \otimes \mathbb{J}^{ij},$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the standard Cartesian basis vectors.

4 Proof of Theorem 1

The existence of the generalized solution to the problem (1.1)–(1.10) for the case $\nu(\theta) \equiv 1$ has been proved in [11]. The non-linear problem is considered in the same way. To derive the desired estimates we consider the equation

$$\begin{aligned} &\frac{d}{dt} \int_\Omega \left(\rho^\varepsilon \alpha_\tau \left(\frac{\partial \mathbf{w}^\varepsilon}{\partial t} \right)^2 + \alpha_\lambda (1-\chi^\varepsilon) \mathbb{D}(x, \mathbf{w}^\varepsilon) : \mathbb{D}(x, \mathbf{w}^\varepsilon) \right. \\ &\quad \left. + c_p^\varepsilon (\theta^\varepsilon)^2 + \alpha_p \chi^\varepsilon (\operatorname{div} \mathbf{w}^\varepsilon)^2 + \alpha_\eta (1-\chi^\varepsilon)(\operatorname{div} \mathbf{w}^\varepsilon)^2 \right) dx \\ &\quad + \int_\Omega \left(\alpha_\mu \nu(\theta) \chi^\varepsilon \mathbb{D}\left(x, \frac{\partial \mathbf{w}^\varepsilon}{\partial t}\right) : \mathbb{D}\left(x, \frac{\partial \mathbf{w}^\varepsilon}{\partial t}\right) + \alpha_\varkappa^\varepsilon |\nabla \theta^\varepsilon|^2 \right) dx = \int_\Omega \mathbf{F} \cdot \frac{\partial \mathbf{w}^\varepsilon}{\partial t}\, dx. \end{aligned} \tag{4.1}$$

We obtain this by multiplying the equation for $\mathbf{w}^\varepsilon$ by $\partial \mathbf{w}^\varepsilon / \partial t$, the equation for θ^ε by θ^ε, summing the result and integrating by parts, using the continuity equations (2.1) and (2.2). Note that all terms on the common interface Γ^ε, the 'solid skeleton–pore space', disappear due to boundary conditions (1.7) and (1.8).

Estimates (2.5) and (2.6) now follow from this energy equation. In fact, if $\tau_0 > 0$, then we just use the Hölder and Gronwall inequalities in (4.1) to get (2.5) and (2.6) together with estimate

$$\int_\Omega \left(\alpha_p \chi^\varepsilon (\operatorname{div} \mathbf{w}^\varepsilon)^2 + \alpha_\eta (1-\chi^\varepsilon)(\operatorname{div} \mathbf{w}^\varepsilon)^2 \right) dx \leqslant C. \tag{4.2}$$

Estimates (2.7) for pressure follow from the continuity equations (2.1) and (2.2) and estimate (4.2).

Estimation of $\mathbf{w}^\varepsilon$ in the case $\tau_0 = 0$ is not simple, and we outline it in more detail. We again use (4.1). The term $\mathbf{F} \cdot \partial \mathbf{w}^\varepsilon / \partial t$ needs additional consideration here. First of all, on the strength of Lemma 3.1, we construct an extension $\mathbf{u}^\varepsilon$ of the function $\mathbf{w}^\varepsilon$ from Ω_s^ε into Ω such that $\mathbf{u}^\varepsilon = \mathbf{w}^\varepsilon$ in Ω_s^ε, $\mathbf{u}^\varepsilon \in W_2^1(\Omega)$ and

$$\|\mathbf{u}^\varepsilon\|_{2,\Omega} \leqslant C\|\nabla \mathbf{u}^\varepsilon\|_{2,\Omega} \leqslant \frac{C}{\sqrt{\alpha_\lambda}}\|(1-\chi^\varepsilon)\sqrt{\alpha_\lambda}\nabla \mathbf{w}^\varepsilon\|_{2,\Omega}.$$

Then we estimate $\|\mathbf{w}^\varepsilon\|_{2,\Omega}$ with the help of Friedrichs–Poincaré's inequality (3.5) for a periodic structure (Lemma 3.2) for the difference $(\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon)$:

$$\|\mathbf{w}^\varepsilon\|_{2,\Omega} \leqslant \|\mathbf{u}^\varepsilon\|_{2,\Omega} + \|\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon\|_{2,\Omega} \leqslant \|\mathbf{u}^\varepsilon\|_{2,\Omega} + C\varepsilon\|\chi^\varepsilon \nabla(\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon)\|_{2,\Omega}$$

$$\leqslant \|\mathbf{u}^\varepsilon\|_{2,\Omega} + C\varepsilon\|\nabla \mathbf{u}^\varepsilon\|_{2,\Omega} + C(\varepsilon\alpha_\mu^{-\frac{1}{2}})\|\chi^\varepsilon\sqrt{\alpha_\mu}\nabla \mathbf{w}^\varepsilon\|_{2,\Omega} \leqslant$$

$$\frac{C}{\sqrt{\alpha_\lambda}}\|(1-\chi^\varepsilon)\sqrt{\alpha_\lambda}\nabla \mathbf{w}^\varepsilon\|_{2,\Omega} + C(\varepsilon\alpha_\mu^{-\frac{1}{2}})\|\chi^\varepsilon\sqrt{\alpha_\mu}\nabla \mathbf{w}^\varepsilon\|_{2,\Omega}.$$

Next we integrate the result with respect to time over the interval $(0, t_0)$, pass the time derivative from $\partial \mathbf{w}^\varepsilon / \partial t$ to $\rho^\varepsilon \mathbf{F}$ and bound all together in the usual way with the help of Hölder and Gronwall's inequalities.

5 Proof of Theorem 2

5.1 Weak and two-scale limits of sequences of displacement and pressures

On the strength of theorem 1, the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\mathbf{w}^\varepsilon\}$ and $\{\nabla\theta^\varepsilon\}$ are bounded in $L^2(\Omega_T)$ uniformly in ε. Hence there exist a subsequence of small parameters $\{\varepsilon > 0\}$ and functions p_f, p_s, $\mathbf{w}$ and θ, such that $p_f^\varepsilon \to p_f$, $p_s^\varepsilon \to p_s$, $\mathbf{w}^\varepsilon \to \mathbf{w}$ weakly in $L^2(\Omega_T)$ and $\theta^\varepsilon \to \theta$ strongly in $L^2(\Omega_T)$ and weakly in $L^2\big(0,T;\overset{\circ}{W}{}_2^1(\Omega)\big)$ as $\varepsilon \searrow 0$.

Relabelling if necessary, we assume that the sequences converge themselves. At the same time

$$\chi^\varepsilon\alpha_\mu \mathbb{D}(x, \mathbf{w}^\varepsilon) \to 0 \tag{5.1}$$

strongly in $L^2(\Omega_T)$ and the sequence $\{\operatorname{div} \mathbf{w}^\varepsilon\}$ converges weakly in $L^2(\Omega_T)$ to $\operatorname{div} \mathbf{w}$ as $\varepsilon \searrow 0$.

Moreover, due to extension Lemma 3.1 there are functions

$$\mathbf{u}^\varepsilon \in L^\infty(0,T;W_2^1(\Omega))$$

such that $\mathbf{u}^\varepsilon = \mathbf{w}^\varepsilon$ in $\Omega_s \times (0,T)$, $\mathbf{u}^\varepsilon = 0$ on the part S_s^ε of the boundary S and

$$\max_{0\leqslant t\leqslant T}\big(\|\mathbf{u}^\varepsilon(t)\|_{2,\Omega} + \|\nabla \mathbf{u}^\varepsilon(t)\|_{2,\Omega}\big) \leqslant C, \tag{5.2}$$

where C does not depend on the small parameter ε.

Estimate (5.2) follows from estimates (2.6) and (3.3).

Lemma 5.1 *Let the sequence* $\{\mathbf{u}^\varepsilon\}$ *satisfy (5.2)and* $\mathbf{u}^\varepsilon = 0$ *on the part* S^ε_s *of the boundary* S. *Then there exist a subsequence of* $\{\varepsilon > 0\}$ *and function*

$$\mathbf{u} \in L^\infty\big(0, T; \overset{\circ}{W}{}^1_2(\Omega)\big),$$

such that $\mathbf{u}^\varepsilon \to \mathbf{u}$ *weakly in* $L^2\big(0, T; W^1_2(\Omega)\big)$ *and strongly in* $L^2(\Omega_T)$ *as* $\varepsilon \searrow 0$.

The proof of the lemma is standard and based on the Friedrichs–Poincaré inequality (3.4).

On the strength of Nguetseng's theorem, there exist functions $P_f(\mathbf{x}, t, \mathbf{y})$, $P_s(\mathbf{x}, t, \mathbf{y})$, $\mathbf{W}(\mathbf{x}, t, \mathbf{y})$, $\Theta(\mathbf{x}, t, \mathbf{y})$ and $\mathbf{U}(\mathbf{x}, t, \mathbf{y})$ which are periodic in $\mathbf{y}$ such that the sequences $\{p^\varepsilon_f\}$, $\{p^\varepsilon_s\}$, $\{\mathbf{w}^\varepsilon\}$, $\{\nabla\theta^\varepsilon\}$ and $\{\nabla\mathbf{u}^\varepsilon\}$ two-scale converge to $P_f(\mathbf{x}, t, \mathbf{y})$, $P_s(\mathbf{x}, t, \mathbf{y})$, $\mathbf{W}(\mathbf{x}, t, \mathbf{y})$, $\nabla\theta(\mathbf{x}, t) + \nabla_y\Theta(\mathbf{x}, t, \mathbf{y})$ and $\nabla\mathbf{u}(\mathbf{x}, t) + \nabla_y\mathbf{U}(\mathbf{x}, t, \mathbf{y})$, respectively.

5.2 Micro- and macroscopic equations I

Lemma 5.2 *For almost all* $\mathbf{x} \in \Omega$ *and for almost all* $\mathbf{y} \in Y$ *the weak and two-scale limits of the sequences* $\{p^\varepsilon_f\}$, $\{p^\varepsilon_s\}$, $\{\mathbf{w}^\varepsilon\}$ *and* $\{\mathbf{u}^\varepsilon\}$ *satisfy the relations*

$$P_f = p_f\frac{\chi}{m}, \qquad P_s = (1-\chi)P_s, \tag{5.3}$$

$$\frac{1}{\eta_0}p_s + (1-m)\operatorname{div}\mathbf{u} + \langle\operatorname{div}_y\mathbf{U}\rangle_{Y_s} = 0, \tag{5.4}$$

$$\frac{1}{\eta_0}P_s + (1-\chi)(\operatorname{div}\mathbf{u} + \operatorname{div}_y\mathbf{U}) = 0, \tag{5.5}$$

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \operatorname{div}\mathbf{w} = 0, \tag{5.6}$$

$$\mathbf{w}(\mathbf{x}, t)\cdot\mathbf{n}(\mathbf{x}) = 0, \quad \mathbf{x} \in S, \tag{5.7}$$

$$\operatorname{div}_y\mathbf{W} = 0, \tag{5.8}$$

$$\mathbf{W} = (1-\chi)\mathbf{u} + \chi\mathbf{W}, \tag{5.9}$$

where $\mathbf{n}(\mathbf{x})$ *is the unit normal vector to* S *at a point* $\mathbf{x} \in S$.

Proof In order to prove (5.4), insert a test function $\psi^\varepsilon = \varepsilon\psi\left(\mathbf{x}, t, \mathbf{x}/\varepsilon\right)$ into (2.3), where $\psi(\mathbf{x}, t, \mathbf{y})$ is an arbitrary 1-periodic function in $\mathbf{y}$ finite on Y_f. Passing to the limit as $\varepsilon \searrow 0$, we get

$$\nabla_y P_f(\mathbf{x}, t, \mathbf{y}) = 0, \quad \mathbf{y} \in Y_f. \tag{5.10}$$

Next, taking the two-scale limit in

$$\chi^\varepsilon p^\varepsilon_s = 0, \qquad (1-\chi^\varepsilon)p^\varepsilon_f = 0$$

we arrive at

$$\chi P_s = 0, \qquad (1-\chi)P_f = 0$$

which along with (5.10) prove (5.4).

Equations (5.5)–(5.8) appear as the results of two-scale limits in (2.1) and (2.2) with appropriate test functions being used. Thus, for example, (5.6) and (5.7) arise, if we consider the sum of (2.1) and (2.2),

$$\frac{1}{\alpha_p} p_f^\varepsilon + \frac{1}{\alpha_\eta} p_s^\varepsilon + \operatorname{div} \mathbf{w}^\varepsilon = 0, \tag{5.11}$$

multiply by an arbitrary function, independent of the 'fast' variable $\mathbf{x}/\varepsilon$, and then pass to the limit as $\varepsilon \searrow 0$. In order to prove (5.8), it is sufficient to consider the two-scale limiting relations in (5.11) as $\varepsilon \searrow 0$ with the test functions $\varepsilon\psi\left(\mathbf{x}/\varepsilon\right) h(\mathbf{x},t)$, where ψ and h are arbitrary smooth functions. In order to prove (5.9) it is sufficient to consider the two-scale limiting relations in

$$(1-\chi^\varepsilon)(\mathbf{w}^\varepsilon - \mathbf{u}^\varepsilon) = 0.$$

□

Lemma 5.3 *Let* $\beta_0(\mathbf{y}) = \beta_{0f}\chi(\mathbf{y}) + \beta_{0s}(1-\chi(\mathbf{y}))$. *Then for almost all* $(\mathbf{x},t) \in \Omega_T$ *the relation*

$$\operatorname{div}_y\left(\lambda_0(1-\chi)(\mathbb{D}(y,\mathbf{U}) + \mathbb{D}(x,\mathbf{u})) - \left(P_s + \frac{1}{m} p_f \chi + \beta_0(\mathbf{y})\theta\right) \cdot \mathbb{I}\right) = 0 \tag{5.12}$$

holds.

Proof Substituting a test function of the form $\psi^\varepsilon = \varepsilon\psi\left(\mathbf{x},t,\mathbf{x}/\varepsilon\right)$, where $\psi(\mathbf{x},t,\mathbf{y})$ is an arbitrary function 1-periodic in $\mathbf{y}$ vanishing on the boundary S, into integral identity (2.3), and passing to the limit as $\varepsilon \searrow 0$, we arrive at (5.12). □

Lemma 5.4 *Let* $\hat{\rho} = m\rho_f + (1-m)\rho_s$ *and* $\hat{\beta}_0 = m\beta_{0f} + (1-m)\beta_{0s}$. *Then functions* $\mathbf{w}^f = \langle \mathbf{W} \rangle_{Y_f}$, $\mathbf{u}$, p_f, p_s *and* θ *satisfy in* Ω_T *the system of macroscopic equations*

$$\begin{aligned} &\tau_0\rho_f \frac{\partial^2 \mathbf{w}^f}{\partial t^2} + \tau_0\rho_s(1-m)\frac{\partial^2 \mathbf{u}}{\partial t^2} - \hat{\rho}\mathbf{F} \\ &= \operatorname{div}(\lambda_0((1-m)\mathbb{D}(x,\mathbf{u}) + \langle \mathbb{D}(y,\mathbf{U})\rangle_{Y_s}) - (p_f + p_s + \hat{\beta}_0\theta)\mathbb{I}), \end{aligned} \tag{5.13}$$

with initial conditions

$$\tau_0\left(\rho_f \mathbf{w}^f + \rho_s(1-m)\mathbf{u}\right) = 0, \tag{5.14}$$

$$\tau_0\left(\rho_f \frac{\partial \mathbf{w}^f}{\partial t} + \rho_s(1-m)\frac{\partial \mathbf{u}}{\partial t} - \hat{\rho}\mathbf{v}_0\right) = 0, \quad \mathbf{x} \in \Omega.$$

Proof Equations (5.13) and initial conditions (5.14) arise as the limit of (2.3) with test functions being independent of ε inΩ_T. □

Lemma 5.5 *If $\mu_1 = \infty$, then the weak limits of $\{\mathbf{u}^\varepsilon\}$ and $\{\mathbf{w}^\varepsilon\}$ coincide and*

$$\mathbf{w}^f = m\mathbf{w}.$$

Proof Let $\Psi(\mathbf{x}, t, \mathbf{y})$ be an arbitrary smooth function periodic in $\mathbf{y}$. The sequence $\{\sigma^\varepsilon_{ij}\}$, where

$$\sigma^\varepsilon_{ij} = \int_\Omega \sqrt{\alpha_\mu}\frac{\partial w^\varepsilon_i}{\partial x_j}(\mathbf{x}, t)\Psi(\mathbf{x}, t, \mathbf{x}/\varepsilon)\,dx, \quad \mathbf{w}^\varepsilon = (w^\varepsilon_1, w^\varepsilon_2, w^\varepsilon_3)$$

is uniformly bounded in ε. Therefore,

$$\int_\Omega \varepsilon\frac{\partial w^\varepsilon_i}{\partial x_j}(\mathbf{x}, t)\Psi(\mathbf{x}, t, \mathbf{x}/\varepsilon)\,dx = \frac{\varepsilon}{\sqrt{\alpha_\lambda}}\sigma^\varepsilon_{ij} \to 0$$

as $\varepsilon \searrow 0$, which is equivalent to

$$\int_\Omega\int_Y W_i(\mathbf{x}, t, \mathbf{y})\frac{\partial \Psi}{\partial y_j}(\mathbf{x}, t, \mathbf{y})\,dx\,dy = 0, \quad \mathbf{W} = (W_1, W_2, W_3),$$

or $\mathbf{W}(\mathbf{x}, t, \mathbf{y}) = \mathbf{w}(\mathbf{x}, t)$. Therefore taking the two-scale limit as $\varepsilon \searrow 0$ in the equality

$$(1 - \chi^\varepsilon)(\mathbf{u}^\varepsilon - \mathbf{w}^\varepsilon) = 0,$$

we arrive at the first statement of the lemma. The last statement follows from the definition of $\mathbf{w}^f$. □

Lemma 5.6 *Let $\mu_1 < \infty$. Then the weak and two-scale limits p_f and $\mathbf{V} = \partial\mathbf{W}/\partial t$ satisfy the microscopic relations*

$$\tau_0\rho_f\frac{\partial \mathbf{V}}{\partial t} = \mu_1\triangle_y\mathbf{V} - \nabla_y R - \nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f\mathbf{F}, \quad \mathbf{y} \in Y_f, \tag{5.15}$$

$$\mathbf{V} = \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{y} \in \gamma \tag{5.16}$$

in the case $\mu_1 > 0$, and relations

$$\tau_0\rho_f\frac{\partial \mathbf{V}}{\partial t} = -\nabla_y R - \nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f\mathbf{F}, \quad \mathbf{y} \in Y_f, \tag{5.17}$$

$$\left(\mathbf{V} - \frac{\partial \mathbf{u}}{\partial t}\right)\cdot\mathbf{n} = 0, \quad \mathbf{y} \in \gamma \tag{5.18}$$

in the case $\mu_1 = 0$.

Differential equations (5.15) and (5.17) are endowed with initial condition

$$\tau_0(\mathbf{V}(\mathbf{y}, 0) - \mathbf{v}_0) = 0, \ \mathbf{y} \in Y_f. \tag{5.19}$$

In (5.18) $\mathbf{n}$ is the unit normal to γ.

Proof The differential equations (5.15), (5.17) and initial conditions (5.19) follow as $\varepsilon \searrow 0$ from the integral equality (2.3) with the test function

$$\psi = \varphi(\mathbf{x}/\varepsilon) \cdot h(\mathbf{x}, t),$$

where φ is solenoidal and finite in Y_s.

Boundary condition (5.16) is a consequence of the two-scale convergence of $\{\sqrt{\alpha_\mu}\nabla \mathbf{w}^\varepsilon\}$ to the function $\sqrt{\mu_1}\nabla_y \mathbf{W}(\mathbf{x}, t, \mathbf{y})$. On the strength of this convergence, the function $\nabla_y \mathbf{W}(\mathbf{x}, t, \mathbf{y})$ is L^2-integrable in Y. The boundary condition (5.18) follows from (5.8) and (5.9). □

Lemma 5.7 *For all* $(\mathbf{x}, t) \in \Omega_T$ *and* $y \in Y$ *strong and two-scale limits* θ *and* Θ *satisfy the microscopic equations*

$$\operatorname{div}_y(\varkappa_0(\mathbf{y})(\nabla\theta + \nabla_y \Theta)) = 0, \tag{5.20}$$

where $\varkappa_0(\mathbf{y}) = \chi(\mathbf{y})\varkappa_{0f} + (1 - \chi(\mathbf{y}))\varkappa_{0s}$.

The proof of this lemma repeats the proof of Lemma 5.3.

Lemma 5.8 *For all* $(\mathbf{x}, t) \in \Omega_T$ *strong and week limits* θ, p_f *and* p_s *satisfy the macroscopic heat equation*

$$\hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*}\frac{\partial p_f}{\partial t} - \frac{\beta_{0s}}{\eta_0}\frac{\partial p_s}{\partial t} = \operatorname{div}(\hat{\varkappa}_0 \nabla\theta + \langle \varkappa_0 \nabla_y \Theta\rangle_Y), \tag{5.21}$$

where $\hat{\varkappa}_0 = \langle \varkappa_0 \rangle_Y$, $\hat{c}_p = mc_{pf} + (1 - m)c_{ps}$ *and initial condition*

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

The proof of this lemma follows that of Lemma 5.4, if we initially exclude the term $\alpha_\theta^\varepsilon \operatorname{div} \mathbf{w}^\varepsilon$ in the integral identity (1.4) using the continuity equations (1.1) and (1.2).

Lemma 5.9 *If the pore space is disconnected, which is the case of isolated pores, then* $\mathbf{u} = \mathbf{w}$.

Proof Indeed, in the case $0 \leqslant \mu_1 < \infty$ the systems of equations (5.8), (5.15) and (5.16) or (5.8), (5.17) and (5.18) have the unique solution $\mathbf{W} = \mathbf{u}$. □

5.4 Homogenized equations I

In this section we derive homogenized equations for the solid component.

Lemma 5.10 *If* $\mu_1 = \infty$ *or the pore space is disconnected then* $\mathbf{w} = \mathbf{u}$ *and strong and weak limits* $\mathbf{u}$, θ, p_f *and* p_s *satisfy in* Ω_T *the initial-boundary value problem*

$$\tau_0 \hat{\rho} \frac{\partial^2 \mathbf{u}}{\partial t^2} + \nabla(p_f + p_s + \hat{\beta}_0 \theta) - \hat{\rho}\mathbf{F} = \tag{5.22}$$

$$\operatorname{div}\big(\lambda_0 \mathbb{A}_0^s : \mathbb{D}(x, \mathbf{u}) + \mathbb{B}_0^s \operatorname{div} \mathbf{u} + \mathbb{B}_1^s(\beta\theta + p_f)\big), \tag{5.23}$$

$$\frac{1}{\eta_0} p_s + \mathbb{C}_0^s : \mathbb{D}(x, \mathbf{u}) + a_0^s \operatorname{div} \mathbf{u} + a_1^s(\beta\theta + p_f) = 0, \tag{5.24}$$

$$\frac{1}{p_*} p_f + \frac{1}{\eta_0} p_s + \operatorname{div} \mathbf{u} = 0, \tag{5.25}$$

where $\beta = m(\beta_{0f} - \beta_{0s})$, the symmetric strictly positively defined constant fourth-rank tensor $\mathbb{A}_0^s$, matrices $\mathbb{C}_0^s$, $\mathbb{B}_0^s$ and $\mathbb{B}_1^s$ and scalars a_0^s and a_1^s are defined below by formulas (5.31)–(5.33).

Differential equations (5.22) are endowed with homogeneous boundary condition

$$\mathbf{u}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in S, \quad t > 0 \tag{5.26}$$

and initial conditions

$$\tau_0 \mathbf{u}(\mathbf{x}, 0) = 0, \qquad \tau_0 \left(\frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, 0) - \mathbf{v}_0(\mathbf{x}) \right) = 0, \quad \mathbf{x} \in \Omega. \tag{5.27}$$

Proof First of all note that due to Lemmas 5.5 and 5.9, $\mathbf{v} = \partial \mathbf{w} / \partial t$.

The homogenized equation (5.22) follows from the macroscopic equation (5.13), after we insert in the relation

$$\lambda_0 \langle \mathbb{D}(y, \mathbf{U}) \rangle_{Y_s} = \lambda_0 \mathbb{A}_1^s : \mathbb{D}(x, \mathbf{u}) + \mathbb{B}_0^s \operatorname{div} \mathbf{u} + \mathbb{B}_1^s \theta + \mathbb{B}_1^s(\mathbf{y})(\beta\theta + p_f).$$

In turn, this follows by virtue of solutions of equations (5.5) and (5.12) on the pattern cell Y_s. In fact, setting

$$\mathbf{U} = \sum_{i,j=1}^{3} \mathbf{U}^{(ij)}(\mathbf{y}) D_{ij} + \mathbf{U}^{(0)}(\mathbf{y}) \operatorname{div} \mathbf{u} + \mathbf{U}^{(1)}(\mathbf{y})(\beta\theta + p_f),$$

$$P_s = \lambda_0 \sum_{i,j=1}^{3} P^{ij}(\mathbf{y}) D_{ij} + P^0(\mathbf{y}) \operatorname{div} \mathbf{u} + P^1(\mathbf{y})(\mathbf{y})(\beta\theta + p_f),$$

where

$$D_{ij}(\mathbf{x}, t) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j}(\mathbf{x}, t) + \frac{\partial u_j}{\partial x_i}(\mathbf{x}, t) \right),$$

we arrive at the following periodic-boundary value problems in Y:

$$\operatorname{div}_y \{ (1 - \chi)(\mathbb{D}(y, \mathbf{U}^{(ij)}) - P^{(ij)} \mathbb{I} + \mathbb{J}^{ij}) = 0,$$

$$(1 - \chi) \left(\frac{\lambda_0}{\eta_0} P^{(ij)} + \operatorname{div}_y \mathbf{U}^{(ij)} \right) = 0; \tag{5.28}$$

$$\operatorname{div}_y \big((1 - \chi)(\lambda_0 \mathbb{D}(y, \mathbf{U}^{(0)}) - P^{(0)} \mathbb{I}) \big) = 0,$$

$$(1 - \chi) \left(\frac{1}{\eta_0} P^{(0)} + \operatorname{div}_y \mathbf{U}^{(0)} + 1 \right) = 0; \tag{5.29}$$

$$\operatorname{div}_y \left((1 - \chi)(\lambda_0 \mathbb{D}(y, \mathbf{U}^{(1)}) - P^{(1)} \mathbb{I}) - \frac{\chi}{m} \mathbb{I} \right) = 0,$$

$$(1 - \chi) \left(\frac{1}{\eta_0} P^{(1)} + \operatorname{div}_y \mathbf{U}^{(1)} \right) = 0. \tag{5.30}$$

On the strength of the assumptions on the geometry of the pattern 'liquid' cell Y_s, problems (5.28)–(5.30) have unique solutions, up to arbitrary constant vectors. In order to discard the arbitrary constant vectors we demand

$$\langle \mathbf{U}^{(ij)} \rangle_{Y_s} = \langle \mathbf{U}^{(0)} \rangle_{Y_s} = \langle \mathbf{U}^{(1)} \rangle_{Y_s} = 0.$$

Thus

$$\mathbb{A}_0^f = (1-m)\mathbb{J} + \mathbb{A}_1^f, \qquad \mathbb{A}_1^f = \sum_{i,j=1}^{3} \langle \mathbb{D}(y, \mathbf{U}^{(ij)}) \rangle_{Y_s} \otimes \mathbb{J}^{ij}, \tag{5.31}$$

$$\mathbb{B}_i^s = \lambda_0 \langle \mathbb{D}(y, \mathbf{U}^{(i)}) \rangle_{Y_s}, \quad i = 0, 1. \tag{5.32}$$

Symmetry and strict positiveness of the tensor $\mathbb{A}_0^s$ have been proved in [8].

Equations (5.24) and (5.25) for the pressures follow from (5.4) and (5.6) and the equality

$$\langle \mathrm{div}_y \mathbf{U} \rangle_{Y_s} = \mathbb{C}_0^s : \mathbb{D}(x, \mathbf{u}) + a_0^s \,\mathrm{div}\, \mathbf{u} + a_1^s(\beta\theta + p_f)$$

with

$$\mathbb{C}_0^s = \sum_{i,j=1}^{3} \langle \mathrm{div}_y \mathbf{U}^{(ij)} \rangle_{Y_s} \mathbb{J}^{ij}, \qquad a_i^s = \langle \mathrm{div}_y \mathbf{U}^{(i)} \rangle_{Y_s}, \quad i = 0, 1. \tag{5.33}$$

Finally note that the initial conditions (5.27) follow from initial conditions (5.14) and (5.19), if we take into account Lemmas 5.5 and 5.9. □

5.5 Homogenized equations II

We complete the proof of Theorem 2 with homogenized equations for the liquid component.

If $\mu_1 < \infty$, then, in the same manner as above, we verify that the strong and weak limits $\mathbf{u}$, θ, p_f and p_s satisfy an initial-boundary value problem like (5.22)–(5.27). The main difference here is that, in general, the weak limit $\mathbf{w}$ of the sequence $\{\mathbf{w}^\varepsilon\}$ differs from $\mathbf{u}$. More precisely, the following statement is true.

Lemma 5.11 *Let $\mu_1 < \infty$. Then the strong and weak limits $\mathbf{u}$, θ, $\mathbf{w}^f$, p_f and p_s of the sequences $\{\mathbf{u}^\varepsilon\}$, $\{\theta^\varepsilon\}$, $\{\chi^\varepsilon \mathbf{w}^\varepsilon\}$, $\{p_f^\varepsilon\}$ and $\{p_s^\varepsilon\}$ satisfy the initial-boundary value problem in the domain Ω_T, consisting of the balance of momentum equation*

$$\begin{aligned} &\tau_0 \rho_s (1-m) \frac{\partial^2 \mathbf{u}}{\partial t^2} + \tau_0 \rho_f \frac{\partial \mathbf{v}}{\partial t} + \nabla(p_f + p_s + \hat{\beta}_0 \theta) - \hat{\rho} \mathbf{F} \\ &= \mathrm{div}\big(\lambda_0 \mathbb{A}_0^s : \mathbb{D}(x, \mathbf{u}) + \mathbb{B}_0^s \,\mathrm{div}\, \mathbf{u} + \mathbb{B}_1^s(\beta\theta + p_f)\big), \end{aligned} \tag{5.34}$$

where $\mathbf{v} = \partial \mathbf{w}^f / \partial t$ and $\mathbb{A}_0^s$, $\mathbb{B}_0^s$ and $\mathbb{B}_1^s$ are the same as in (5.22), the continuity equation (5.24) for the solid component and continuity equation

$$\frac{1}{p_*} \frac{\partial p_f}{\partial t} + \frac{1}{\eta_0} \frac{\partial p_s}{\partial t} + \mathrm{div}\, \mathbf{v} = (m-1)\,\mathrm{div} \frac{\partial \mathbf{u}}{\partial t} \tag{5.35}$$

for the liquid component, the relation

$$\mathbf{v} = m\frac{\partial \mathbf{u}}{\partial t} + \int_0^t \mathbb{B}_1(t-\tau)\cdot \mathbf{h}(\mathbf{x},\tau)\,d\tau, \tag{5.36}$$

$$\mathbf{h} = -\nabla\Big(\frac{1}{m}p_f + \beta_{0f}\theta\Big) + \rho_f\mathbf{F} - \tau_0\rho_f\frac{\partial^2 \mathbf{u}}{\partial t^2}$$

in the case of $\tau_0 > 0$ *and* $\mu_1 > 0$*, or Darcy's law in the form*

$$\mathbf{v} = m\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\mu_1}\mathbb{B}_2\cdot\Big(-\nabla\Big(\frac{1}{m}p_f + \beta_{0f}\theta\Big) + \rho_f\mathbf{F}\Big) \tag{5.37}$$

in the case of $\tau_0 = 0$ *or, finally, the balance of momentum equation for the liquid component in the form*

$$\tau_0\rho_f\frac{\partial \mathbf{v}}{\partial t} = \tau_0\rho_f\mathbb{B}_3\cdot\frac{\partial^2 \mathbf{u}}{\partial t^2} + (m\mathbb{I} - \mathbb{B}_3)\cdot\Big(-\nabla\Big(\frac{1}{m}p_f + \beta_{0f}\theta\Big) + \rho_f\mathbf{F}\Big) \tag{5.38}$$

in the case of $\mu_1 = 0$ *for the liquid component.*

The problem is supplemented by boundary and initial conditions (5.26) and (5.27) for displacement $\mathbf{u}$ *of the rigid component and by the initial and boundary conditions*

$$\tau_0(\mathbf{v}(\mathbf{x},0) - m\mathbf{v}_0(\mathbf{x})) = 0, \quad \mathbf{x}\in\Omega, \tag{5.39}$$

$$\mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = 0,\ \mathbf{x}\in S, \quad t>0 \tag{5.40}$$

for the velocity $\mathbf{v}$ *of the liquid component. In (5.36)–(5.40)* $\mathbf{n}(\mathbf{x})$ *is the unit normal vector to* S *at a point* $\mathbf{x}\in S$*, and matrix* $\mathbb{B}_1(t)$*, symmetric strictly positively defined matrices* $\mathbb{B}_2$ *and* $(m\mathbb{I} - \mathbb{B}_3)$ *are defined below by formulas (5.42), (5.44) and (5.46).*

Proof The boundary condition (5.40) follows from (5.7), the equation

$$\mathbf{w} = \mathbf{w}^f + (1-m)\mathbf{u},$$

and the homogeneous boundary condition for $\mathbf{u}$.

The same equation and (5.6) imply (5.36). The homogenized equation for balance of momentum (5.34) is derived exactly as before. Therefore we omit the proofs now and focus only on the derivation of the homogenized equation for the balance of momentum for the liquid velocity $\mathbf{v} = \partial\mathbf{w}^f/\partial t$.

(a) Let $\tau_0 > 0$ and $\mu_1 > 0$. To solve the system of microscopic equations (5.8), (5.15) and (5.16), provided with initial data (5.19), first of all we represent the solutions $\mathbf{V}(\mathbf{x},t,\mathbf{y})$ and $R(\mathbf{x},t,\mathbf{y})$ by

$$\mathbf{V}(\mathbf{x},t,\mathbf{y}) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t) + \int_0^t \sum_{i=1}^3 \mathbf{V}_0^i(\mathbf{y},t-\tau)(\mathbf{e}_i\cdot\mathbf{h}(\mathbf{x},\tau))\,d\tau,$$

$$R(\mathbf{x},t,\mathbf{y}) = \int_0^t \sum_{i=1}^{3} R_0^i(\mathbf{y},t-\tau)(\mathbf{e}_i \cdot \mathbf{h}(\mathbf{x},\tau))\,d\tau,$$

in which functions $\mathbf{V}_0^i(\mathbf{y},t)$ and $R_0^i(\mathbf{y},t)$ are defined by the periodic initial-boundary value problem

$$\tau_0\rho_f\frac{\partial \mathbf{V}_0^i}{\partial \tau} - \mu_1\triangle\mathbf{V}_0^i + \nabla R_0^i = 0, \qquad \mathrm{div}_y\mathbf{V}_0^i = 0, \quad \mathbf{y}\in Y_f,\ t>0;$$

$$\mathbf{V}_0^i = 0, \quad \mathbf{y}\in\gamma, \quad \tau>0; \tag{5.41}$$

$$\mathbf{V}_0^i(y,0) = \frac{1}{\tau_0\rho_f}\mathbf{e}_i, \quad \mathbf{y}\in Y_s.$$

In (5.41) $\mathbf{e}_i$ is the standard Cartesian basis vector. Therefore

$$\mathbb{B}_1(t) = \sum_{i=1}^{3}\langle\mathbf{V}_0^i\rangle_{Y_f}(t)\otimes\mathbf{e}_i. \tag{5.42}$$

Note that the differential equations in (5.41) are understood in the sense of distributions. Compatibility conditions do not apply and the time derivative of the function $\mathbf{V}_0^i$ at $t=0$ on the boundary γ is unbounded.

(b) If $\tau_0 = 0$ and $\mu_1 > 0$, then the solution of the system of microscopic equations (5.8), (5.17) and (5.18) is given by the formula

$$\mathbf{V} = \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\mu_1}\mathbb{B}_2^f(\mathbf{y})\cdot\Big(-\nabla\Big(\frac{1}{m}p_f + \beta_{0f}\theta\Big) + \rho_f\mathbf{F}\Big),$$

in which

$$\mathbb{B}_2^f(\mathbf{y}) = \sum_{i=1}^{3}\mathbf{V}_1^i(\mathbf{y})\otimes\mathbf{e}_i,$$

and the functions $\mathbf{V}_1^i$ are defined by the periodic-boundary value problem

$$-\triangle\mathbf{V}_1^i + \nabla R_1^i = \mathbf{e}_i, \quad \mathrm{div}_y\mathbf{V}_1^i = 0, \quad \mathbf{y}\in Y_f; \qquad \mathbf{V}_1^i = 0, \quad \mathbf{y}\in\gamma. \tag{5.43}$$

Thus

$$\mathbb{B}_2 = \langle\mathbb{B}_2^f((\mathbf{y})\rangle_{Y_f}. \tag{5.44}$$

The matrix $\mathbb{B}_2$ is symmetric and strictly positively defined [14, Chap. 8].

(c) If $\mu_1 = 0$ then in the process of solving the system (5.8), (5.17) and (5.18) we firstly find the pressure $R(\mathbf{x},t,\mathbf{y})$ through solving the Neumann problem for Laplace's equation for Y_f in the form

$$R(\mathbf{x},t,\mathbf{y}) = \sum_{i=1}^{3} R_i(\mathbf{y})\mathbf{e}_i\cdot\mathbf{h}(\mathbf{x},t),$$

where $R^i(\mathbf{y})$ is the solution of the problem

$$\triangle_y R_i = 0, \quad \mathbf{y}\in Y_f; \qquad \nabla_y R_i\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{e}_i, \quad \mathbf{y}\in\gamma. \tag{5.45}$$

Formula (5.38) appears as the result of integration of (5.17) over the domain Y_f and

$$\mathbb{B}_3 = \sum_{i=1}^{3} \langle \nabla R_i(\mathbf{y}) \rangle_{Y_f} \otimes \mathbf{e}_i, \tag{5.46}$$

where the matrix $\mathbb{B} = (m\mathbb{I} - \mathbb{B}_3)$ is symmetric and positive definite. In fact, let $\tilde{R} = \sum_{i=1}^{3} R_i \xi_i$ for any unit vector ξ. Then

$$(\mathbb{B} \cdot \xi) \cdot \xi = \langle |\xi - \nabla \tilde{R}|^2 \rangle_{Y_f} > 0.$$

On the strength of our assumptions on the geometry of pattern 'solid' cell Y_f, problems (5.41) and (5.43) have unique solutions and problems (5.45) have unique solutions up to arbitrary constants. □

We complete the proof of theorem with

Lemma 5.12 *For almost all $(\mathbf{x}, t) \in \Omega_T$ strong and weak limits θ, p_f and p_s satisfy the homogenized heat equation*

$$\hat{c}_p \frac{\partial \theta}{\partial t} - \frac{\beta_{0f}}{p_*} \frac{\partial p_f}{\partial t} - \frac{\beta_{0s}}{\eta_0} \frac{\partial p_s}{\partial t} = \mathrm{div}\big(\mathbb{B}^\theta \cdot \nabla \theta\big), \tag{5.47}$$

and initial and boundary conditions

$$\theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad \theta(\mathbf{x}, t) = 0, \quad \mathbf{x} \in S,\ t > 0. \tag{5.48}$$

The symmetric and strictly positively defined matrix $\mathbb{B}^\theta$ is given below by formula (5.49).

Proof The homogenized heat equation (5.47) is a macroscopic equation (5.21), where the expression $\langle \varkappa_0 \nabla_y \Theta \rangle_Y$ is substituted by $\mathbb{B}_0^\theta \cdot \nabla \theta$. The last expression is defined by a solution of the microscopic heat equation (5.20) in the form

$$\Theta(\mathbf{x}, t, \mathbf{y}) = \sum_{i=1}^{3} \Theta_i(\mathbf{y}) \frac{\partial \theta}{\partial x_i}(\mathbf{x}, t),$$

where Θ_i, $i = 1, 2, 3$, are periodic solutions of the problems

$$\mathrm{div}_y(\varkappa_0(\nabla_y \Theta_i + \mathbf{e}_i)) = 0$$

in the domain Y. Thus

$$\mathbb{B}^\theta = \hat{\varkappa}_0 \mathbb{I} + \mathbb{B}_0^\theta, \quad \mathbb{B}_0^\theta = \sum_{i=1}^{3} \nabla_y \langle \Theta_i \rangle_Y \otimes \mathbf{e}_i. \tag{5.49}$$

6 Proof of Theorem 3

6.1 Weak and two-scale limits of sequences of displacement and pressures

On the strength of Theorems 1 and 4 we conclude that the sequences $\{\chi^\varepsilon \mathbf{w}^\varepsilon\}$ and $\{p_f^\varepsilon\}$ two-scale converge to $\chi(\mathbf{y})\mathbf{W}(\mathbf{x},t,\mathbf{y})$ and $p_f(\mathbf{x},t)\chi(\mathbf{y})$ and weakly converge in $L^2(\Omega_T)$ to $\mathbf{w}^f$ and p_f, respectively, and the sequence $\{\theta^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$ and weakly in $L^2(0,T;\overset{\circ}{W}{}_2^1(\Omega))$ to the function θ. A sequence $\{\mathbf{u}^\varepsilon(\mathbf{x},t)\}$, where $\mathbf{u}^\varepsilon(\mathbf{x},t)$ is an extension of $\mathbf{w}^\varepsilon(\mathbf{x},t)$ from the domain Ω_s^ε into domain Ω, strongly converges in $L^2(\Omega_T)$ and weakly in $L^2((0,T);W_2^1(\Omega))$ to zero.

6.2 Homogenized equations

As in the proof of Theorem 2, we construct a closed system of equations for the velocity $\mathbf{v} = \partial \mathbf{w}^f/\partial t$ in the liquid component, the pressure p_f and for the temperature θ.

(a) Let $\tau_0 > 0$ and $\mu_1 > 0$. Then the the system of microscopic equations (5.8), (5.15) and (5.16), provided with initial data (5.19), has the form

$$\frac{\tau_0\rho_f}{\mu_1 \nu(\theta)}\frac{\partial \mathbf{V}}{\partial t} = \triangle_y \mathbf{V} - \nabla_y R + \mathbf{z}, \qquad \mathrm{div}_y \mathbf{V} = 0, \quad \mathbf{y} \in Y_f; \tag{6.1}$$

$$\mathbf{V}(\mathbf{y},t) = 0, \quad \mathbf{y} \in \gamma,\ t > 0; \qquad \mathbf{V}(\mathbf{y},0) = v_0(\mathbf{x}), \quad \mathbf{y} \in Y_f, \tag{6.2}$$

where

$$\mathbf{z} = -\frac{1}{\mu_1\nu(\theta)}\left(\nabla\left(\frac{1}{m}p_f + \beta_{0f}\theta\right) + \rho_f \mathbf{F}\right).$$

To solve this problem (6.1), (6.2) we introduce a new time $\tau = \tau(\mathbf{x},t)$ by the formula

$$\frac{\partial \tau}{\partial t} = \frac{\mu_1 \nu(\theta)}{\tau_0 \rho_f}, \quad \tau(\mathbf{x},0) = 0.$$

Then the problem (6.1), (6.2) reduces to the previously solved problem (5.8), (5.15), (5.16) and (5.19) with $\nu(\theta) \equiv 1$ for the function

$$\tilde{\mathbf{V}}(\mathbf{x},\tau,\mathbf{y}) = \mathbf{V}(\mathbf{x},t,\mathbf{y}) - \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x},t)$$

and

$$\langle \tilde{\mathbf{V}}\rangle_{Y_f}(\mathbf{x},\tau) = \tilde{\mathbf{v}}(\mathbf{x},\tau) = \int_0^\tau \mathbb{B}_1(\tau-\xi)\cdot \tilde{\mathbf{z}}(\mathbf{x},\xi)\,d\xi,$$

where

$$\tilde{\mathbf{z}}\big(\mathbf{x},\tau(\mathbf{x},t)\big) \equiv \mathbf{z}(\mathbf{x},t).$$

After that the liquid velocity is defined through the identity

$$\mathbf{v}(\mathbf{x},t) = m\mathbf{v}_0(\mathbf{x}) + \tilde{\mathbf{v}}(\mathbf{x},\tau(\mathbf{x},t)).$$

The next step is a derivation of the homogenized heat equation. The only difference here compared with the previous case is in the term

$$(1-\chi^{\varepsilon}(\mathbf{x}))\alpha_{\theta s}\operatorname{div}\mathbf{w}^{\varepsilon},$$

which converges strongly in $L^2(\Omega_T)$ to zero as $\varepsilon \searrow 0$. Therefore

$$\hat{c}_p\frac{\partial\theta}{\partial t}-\frac{\beta_{0f}}{p_*}\frac{\partial p_f}{\partial t}=\operatorname{div}(\mathbb{B}^{\theta}\cdot\nabla\theta).$$

Finally, to get a continuity equation we rewrite (2.1) in the form

$$\frac{1}{\alpha_p}p_f^{\varepsilon}+\operatorname{div}\mathbf{w}^{\varepsilon}=(1-\chi^{\varepsilon})\operatorname{div}\mathbf{u}^{\varepsilon},$$

which implies

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t}+\operatorname{div}\mathbf{v}=0.$$

The problem is endowed with corresponding boundary and initial conditions.

(b) For the case $\tau_0=0$ and $\mu_1>0$ we just repeat the proof of Theorem 2. Taking into account the equality $\mathbf{u}=0$ we get

$$\mathbf{v}=\frac{1}{\mu_1}\mathbb{B}_2\cdot\Big(-\nabla\Big(\frac{1}{m}p_f+\beta_{0f}\theta\Big)+\rho_f\mathbf{F}\Big),$$

$$\frac{1}{p_*}\frac{\partial p_f}{\partial t}+\operatorname{div}\mathbf{v}=0,$$

$$\hat{c}_p\frac{\partial\theta}{\partial t}-\frac{\beta_{0f}}{p_*}\frac{\partial p_f}{\partial t}=\operatorname{div}(\mathbb{B}^{\theta}\cdot\nabla\theta).$$

The problem is supplemented with homogeneous boundary conditions for the velocity $\mathbf{v}$ and temperature θ, homogeneous initial condition for the pressure p_f and initial conditions for the temperature.

(c) Finally, if $\mu_1=0$, we again repeat the proof of Theorem 2 taking into account the equality $\mathbf{u}=0$.

7 Conclusions

In the present publication we have derived completely new systems of anisothermic liquid filtration and anisothermic acoustic equations and showed (Theorem 2) that for the case *(I)* ($\mu_0=0$, $0<\lambda_0<\infty$) the limiting regime is a two-velocity continuum. This is described by the Biot system of equations of poro-elasticity coupled with a corresponding heat equation ($0<\mu_1<\infty$, $\tau_0=0$) or a similar system, consisting of anisotropic Lamé equations for a thermoelastic solid coupled with acoustic equations for the thermofluid ($\mu_1=0$), or a one-velocity continuum, described by anisotropic Lam'e system of equations coupled with a corresponding heat equation ($\mu_1=\infty$, or isolated pores for any criteria). Thus, for some situations a one-velocity continuum before homogenization becomes a two-velocity continuum after homogenization. It appears as a result of the different smoothness of the solution in the solid and in the liquid components:

$$\int_{\Omega} \alpha_{\mu}(\varepsilon)\chi^{\varepsilon}|\nabla \mathbf{w}^{\varepsilon}|^{2}\,dx \leqslant C_{0}, \quad \int_{\Omega} \alpha_{\lambda}(\varepsilon)(1-\chi^{\varepsilon})|\nabla \mathbf{w}^{\varepsilon}|^{2})\,dx \leqslant C_{0},$$

where C_0 is a constant independent of the small parameter ε. To preserve the best properties of the solution we must use the well-known extension lemma [1, 6] and extend the solution from the solid part to the liquid one. At this stage the condition on μ_1, becomes crucial. Namely, if $\mu_0 = 0$ and $\lambda_0 < \infty$, then the limiting (homogenized) system describes a two-velocity continuum if $\mu_1 < \infty$ and a one-velocity continuum if $\mu_1 = \infty$. The last case occurs because the sequence $\{\mathbf{w}^{\varepsilon}\}$ two-scale converges to a function independent of the fast variable. This statement easily follows from Nguetseng's theorem.

For the case *(II)* ($\lambda_0 = \infty$, $0 \leqslant \mu_1 < \infty$) all situations are covered by Theorem 3. The limiting regimes here are described by Darcy's system of equations of filtration of slightly compressible viscous thermofluid ($0 < \mu_1 < \infty$ and $\tau_0 = 0$), or two different systems of acoustic equations ($0 < \mu_1 < \infty$ and $\tau_0 > 0$, or $\mu_1 = 0$ and $\tau_0 > 0$) for the liquid component, coupled with a corresponding heat equation.

The presence of a non-linear viscosity $v(\theta)$ essentially changes the form of the homogenized equations if $\tau_0 > 0$ but does not change the expected form if $\tau_0 = 0$. To simplify the paper we show it just for case *(II)*.

Finally note that in practice to solve a real physical problem in, for example, non-isothermal filtration, one doesn't want to have to carry out a limiting procedure but instead has to find a simple and trustable mathematical model describing a process. But there is only an exact mathematical model (1.1)–(1.10) (sufficiently trustable), given physical constants (densities, viscosities, etc.), the characteristic size L of the physical domain in consideration and the characteristic time τ of the physical process. The small parameter ε and dimensionless quantities α_{μ}, α_{λ}, $\alpha_{p}, \ldots$ are functions of these. Changing the values of L and τ within reasonable limits one may find some rules for the behaviour of the dimensionless quantities as the small parameter tends to zero. All possible limits of these quantities are described by conditions on μ_0, λ_0, $\mu_1, \ldots$ and as we have mentioned above, each homogenized system corresponds to the given combination of these. Thus, for a given physical situation there exists some combination of dimensionless criteria, that would suggest the choice of the form of the homogenized system for the exact mathematical model. Therefore the finding of all possible homogenized systems is very important both from mathematical and practical points of view.

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