On the Relationship between the Integrated Cosine Function and the Operator Bessel Function

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The weakening of conditions imposed on the solution operators of the Cauchy problem for abstract first- and second-order differential equations has led (see [1–3]) to the notion of integrated semigroup and integrated cosine function.

In the present paper, we derive formulas relating the integrated cosine function to the solution operator $Y_k(t)$ of the Cauchy problem

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \qquad t > 0, \tag{1}$$

$$u(0) = u_0, \qquad u'(0) = 0, \tag{2}$$

for the Euler–Poisson–Darboux equation in a Banach space E. (Here $k > 0$ is a parameter.)

The operator function $Y_k(t)$ was introduced in [4] and named the operator Bessel function. The set of operators A for which problem (1), (2) is uniformly well posed will be denoted by G_k. Thus if $A \in G_k$, then problem (1), (2) has a unique solution, which continuously depends on the initial data; moreover, $u(t) = Y_k(t)u_0$, $u_0 \in D(A)$, and

$$\|Y_k(t)\| \le Me^{\omega t}, \qquad M \ge 1, \qquad \omega \ge 0. \tag{3}$$

Note that the condition for problem (1), (2) to be uniformly well posed and the properties of the operator Bessel function $Y_k(t)$ were given in [4].

Next, recall the definition of integrated cosine function.

Definition 1. Let $\alpha > 0$. A one-parameter family $C_\alpha(t)$, $t \ge 0$, of bounded linear operators is called an *α-times integrated cosine function* if the following conditions are satisfied:

1.

$$2\Gamma(\alpha)C_\alpha(t)C_\alpha(s) = \int\limits_t^{t+s}(t+s-r)^{\alpha-1}C_\alpha(r)dr - \int\limits_0^s(t+s-r)^{\alpha-1}C_\alpha(r)dr$$
$$+ \int\limits_{t-s}^{t}(r-t+s)^{\alpha-1}C_\alpha(r)dr + \int\limits_0^s(r+t-s)^{\alpha-1}C_\alpha(r)dr, \qquad t > s > 0.$$

2. $C_\alpha(0) = 0$.

3. $C_\alpha(t)x$ is a continuous function of $t \ge 0$ for each $x \in E$.

4. There exist constants $M > 0$ and $\omega \ge 0$ such that

$$\|C_\alpha(t)\| \le Me^{\omega t}, \qquad t \ge 0. \tag{4}$$

The *generator* A of an integrated cosine function $C_\alpha(t)$ is defined as follows: the domain $D(A)$ is the set of elements $x \in E$ such that there exists an element $y \in E$ satisfying the relation

$$C_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x = \int\limits_0^t (t-r)C_\alpha(r)y\,dr, \qquad t \geq 0, \tag{5}$$

where $\Gamma(\cdot)$ is the Euler gamma function; in this case, we set $Ax = y$.

Theorem 1. *Let $\alpha > 1$, let an operator A be the generator of an α-times integrated cosine function $C_\alpha(t)$, and let $u_0 \in D(A)$. Then problem (1), (2) is uniformly well posed (i.e., $A \in G_k$), and the corresponding operator Bessel function can be represented in the form*

$$Y_{2\alpha}(t)u_0 = \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}t^\alpha}\left(C_\alpha(t)u_0 - \frac{1}{t}\int\limits_0^t P'_{\alpha-1}\left(\frac{s}{t}\right)C_\alpha(s)u_0 ds\right), \tag{6}$$

where $P_\nu(\cdot)$ is a spherical Legendre function.

Proof. Formula (6) can be obtained heuristically as follows. Consider an operator cosine function $C(t)$ and set

$$IC(t) = \int\limits_0^t C(\tau)d\tau.$$

Then, by the formula [4] for a parameter shift in Eq. (1),

$$Y_k(t)u_0 = \frac{2t^{1-k}\Gamma(k/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)}\int\limits_0^t \left(t^2-s^2\right)^{m-1}\frac{d^m\left(I^mC(s)u_0\right)}{ds^m}ds \tag{7}$$

for $k = 2m$, $m \in N$, or, after simple transformations,

$$\begin{aligned} Y_k(t)u_0 &= \frac{2(-1)^{m-1}t^{1-k}\Gamma(k/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)}\int\limits_0^t \frac{d^{m-1}\left(\left(t^2-s^2\right)^{m-1}\right)}{ds^{m-1}}\frac{d\left(I^mC(s)u_0\right)}{ds}ds \\ &= \frac{2^m\Gamma(m+1/2)}{\sqrt{\pi}\,t^m}\int\limits_0^t P_{m-1}\left(\frac{s}{t}\right)\frac{d\left(I^mC(s)u_0\right)}{ds}ds \\ &= \frac{2^m\Gamma(m+1/2)}{\sqrt{\pi}\,t^m}\left(I^mC(t)u_0 - \frac{1}{t}\int\limits_0^t P'_{m-1}\left(\frac{s}{t}\right)I^mC(s)u_0 ds\right), \end{aligned} \tag{8}$$

where $P_{m-1}(\cdot)$ is a Legendre polynomial.

In (8), we replace $m \in N$ by $\alpha > 1$, $I^mC(t)$ by $C_\alpha(t)$, and the Legendre polynomial $P_{m-1}(\cdot)$ by the spherical Legendre function $P_{\alpha-1}(\cdot)$. Let us show that the function $Y_{2\alpha}(t)u_0$ defined in (6) is a solution of problem (1), (2) for $k = 2\alpha$.

Let us compute the first and second derivatives of $Y_{2\alpha}(t)u_0$:

$$Y'_{2\alpha}(t)u_0 = \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}}\left(-\frac{\alpha+P'_{\alpha-1}(1)}{t^{\alpha+1}}C_\alpha(t)u_0 + \frac{1}{t^\alpha}C'_\alpha(t)u_0\right.$$

$$\left.+\frac{1+\alpha}{t^{\alpha+2}}\int\limits_0^t P'_{\alpha-1}\left(\frac{s}{t}\right)C_\alpha(s)u_0ds + \frac{1}{t^{\alpha+3}}\int\limits_0^t P''_{\alpha-1}\left(\frac{s}{t}\right)sC_\alpha(s)u_0ds\right),$$

$$Y''_{2\alpha}(t)u_0 = \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}}\left(\frac{P''_{\alpha-1}(1)+2(\alpha+1)P'_{\alpha-1}(1)+\alpha^2+\alpha}{t^{\alpha+2}}C_\alpha(t)u_0\right.$$

$$-\frac{2P'_{\alpha-1}(1)+2\alpha}{t^{\alpha+1}}C'_\alpha(t)u_0 + \frac{1}{t^\alpha}C''_\alpha(t)u_0 - \frac{(\alpha+1)(\alpha+2)}{t^{\alpha+3}}\int\limits_0^t P'_{\alpha-1}\left(\frac{s}{t}\right)C_\alpha(s)u_0ds$$

$$\left.-\frac{2\alpha+4}{t^{\alpha+4}}\int\limits_0^t P''_{\alpha-1}\left(\frac{s}{t}\right)sC_\alpha(s)u_0ds - \frac{1}{t^{\alpha+5}}\int\limits_0^t P'''_{\alpha-1}\left(\frac{s}{t}\right)s^2C_\alpha(s)u_0ds\right).$$

Then, after integration by parts, we obtain

$$\begin{aligned}&Y''_{2\alpha}(t)u_0 + \frac{2\alpha}{t}Y'_{2\alpha}(t)u_0\\
&= \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}}\left(\frac{\alpha-\alpha^2}{t^{\alpha+2}}C_\alpha(t)u_0 - \frac{P'_{\alpha-1}(1)}{t^{\alpha+1}}C'_\alpha(t)u_0 + \frac{1}{\Gamma(\alpha-1)t^2}u_0 + \frac{1}{t^\alpha}C_\alpha(t)Au_0\right.\\
&\left.+\frac{\alpha^2-\alpha}{t^{\alpha+3}}\int\limits_0^t P'_{\alpha-1}\left(\frac{s}{t}\right)C_\alpha(s)u_0ds + \frac{1}{t^{\alpha+2}}\int\limits_0^t\left(\frac{s^2}{t^2}P''_{\alpha-1}\left(\frac{s}{t}\right)+\frac{2s}{t}P'_{\alpha-1}\left(\frac{s}{t}\right)\right)C'_\alpha(s)u_0ds\right).\end{aligned} \tag{9}$$

By [5, p. 206], the spherical Legendre function $P_{\alpha-1}(s)$ is a solution of the equation

$$\left(1-s^2\right)w''(s) - 2sw'(s) + \alpha(\alpha-1)w(s) = 0;$$

consequently, the function $P_{\alpha-1}(s/t)$ satisfies the relation

$$\frac{s^2}{t^2}w''\left(\frac{s}{t}\right) + \frac{2s}{t}w'\left(\frac{s}{t}\right) = w''\left(\frac{s}{t}\right) + \alpha(\alpha-1)w\left(\frac{s}{t}\right). \tag{10}$$

By taking into account (10) and by integrating by parts, from (9), we obtain

$$\begin{aligned}Y''_{2\alpha}(t)u_0 + \frac{2\alpha}{t}Y'_{2\alpha}(t)u_0 &= \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}}\left(\frac{1}{\Gamma(\alpha-1)t^2}u_0 + \frac{1}{t^\alpha}C_\alpha(t)Au_0\right.\\
&\left.-\frac{1}{t^{\alpha+1}}\int\limits_0^t P'_{\alpha-1}\left(\frac{s}{t}\right)\left(\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}u_0 + C_\alpha(s)Au_0\right)ds\right)\\
&= \frac{2^\alpha\Gamma(\alpha+1)}{\sqrt{\pi}}\left(\frac{1}{t^\alpha}C_\alpha(t)Au_0 - \frac{1}{t^{\alpha+1}}\int\limits_0^t P'_{\alpha-1}\left(\frac{s}{t}\right)C_\alpha(s)Au_0ds\right.\\
&\left.+\frac{u_0}{\Gamma(\alpha-1)t^2} - \frac{u_0}{\Gamma(\alpha-1)t^{\alpha+1}}\int\limits_0^t s^{\alpha-2}P'_{\alpha-1}\left(\frac{s}{t}\right)ds\right).\end{aligned} \tag{11}$$

We use [6, Eq. 1.12.1.15] to compute the integral

$$\begin{aligned}
\int_0^t s^{\alpha-2}P'_{\alpha-1}\left(\frac{s}{t}\right)ds &= t^{\alpha-1}\int_0^t \tau^{\alpha-2}P'_{\alpha-1}(\tau)d\tau \\
&= t^{\alpha-1}\left(\tau^{\alpha-2}P_{\alpha-1}(\tau) - (\alpha-2)\int_0^t \tau^{\alpha-3}P_{\alpha-1}(\tau)d\tau\right) \\
&= t^{\alpha-1}\left(\tau^{\alpha-2}P_{\alpha-1}(\tau) - \frac{\tau^{\alpha-2}}{\alpha-1}\left(\alpha\tau P_\alpha(\tau) - \left((2\alpha-1)\tau^2-\alpha+1\right)P_{\alpha-1}(\tau)\right)\right) \\
&= \frac{t^{\alpha-1}}{\alpha-1}\left((2\alpha-1)\tau^\alpha P_{\alpha-1}(\tau) - \alpha\tau^{\alpha-1}P_\alpha(\tau)\right);
\end{aligned}$$

hence

$$\int_0^t s^{\alpha-2}P'_{\alpha-1}\left(\frac{s}{t}\right)ds = t^{\alpha-1}. \tag{12}$$

It follows from (11), (12), and (6) that

$$Y''_{2\alpha}(t)u_0 + \frac{2\alpha}{t}Y'_{2\alpha}(t)u_0 = AY_{2\alpha}(t)u_0.$$

Therefore, the function $Y_{2\alpha}(t)u_0$ is a solution of Eq. (1).

Note that the representation (6) (after the change of variables $s = t\tau$ in the integral), together with (4), implies the estimate

$$\|Y_{2\alpha}(t)\| \le M_1 e^{\omega t}. \tag{13}$$

To show that the function $Y_{2\alpha}(t)u_0$ satisfies the initial condition (2), we use relation (5) and the integral 2.17.1.4 in [6] and rewrite the expression (6) in the form

$$\begin{aligned}
Y_{2\alpha}(t)u_0 &= \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}\,t^\alpha}\int_0^t P_{\alpha-1}\left(\frac{s}{t}\right)C'_\alpha(s)u_0ds \\
&= \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}\,t^\alpha}\int_0^t P_{\alpha-1}\left(\frac{s}{t}\right)\left(\frac{s^{\alpha-1}u_0}{\Gamma(\alpha)} + \int_0^s C_\alpha(\varrho)Au_0d\varrho\right)ds \\
&= \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}\,t^\alpha}\left(\frac{1}{\Gamma(\alpha)}\int_0^1 \tau^{\alpha-1}P_{\alpha-1}(\tau)u_0d\tau + \int_0^t P_{\alpha-1}\left(\frac{s}{t}\right)ds\int_0^s C_\alpha(\varrho)Au_0d\varrho\right) \\
&= u_0 + \frac{2^\alpha\Gamma(\alpha+1/2)}{\sqrt{\pi}\,t^{\alpha-1}}\int_0^1 P_{\alpha-1}(\tau)d\tau\int_0^{t\tau} C_\alpha(\varrho)Au_0d\varrho.
\end{aligned} \tag{14}$$

Now the desired assertion follows from (5), since the last term in (14) is of the order of t^2 as $t \to 0$.

We prove the uniqueness of the solution of problem (1), (2) by contradiction. Let $u_1(t)$ and $u_2(t)$ be two solutions of problem (1), (2). Consider the function $w(t,s) = f\left(Y_{2\alpha}(s)\left(u_1(t)-u_2(t)\right)\right)$ of two variables $t, s \ge 0$, where f belongs to the adjoint space E^*. Obviously, it satisfies the equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{2\alpha}{t}\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial s^2} + \frac{2\alpha}{s}\frac{\partial w}{\partial s}, \qquad t, s > 0,$$

and the conditions

$$w(0,s) = \frac{\partial w(0,s)}{\partial t} = \frac{\partial w(t,0)}{\partial s} = 0.$$

By using the change of variables $t_1 = (t+s)^2/4$, $s_1 = (t-s)^2/4$, one can reduce [7, Sec. 5, item 3] the last problem to the problem whose uniqueness in the class of twice continuously differentiable functions for $t, s \geq 0$ was proved in [7, Sec. 5, item 2]. Moreover, the desired uniqueness is also contained in Theorem 6.1 in [8], where even a more general equation was considered.

Consequently, $w(t,s) \equiv 0$, and since the functional $f \in E^*$ is arbitrary, for $s = 0$, we obtain the relation $u_1(t) \equiv u_2(t)$, and the proof of the uniqueness is complete.

Thus the operator function $Y_{2\alpha}(t)$ satisfies inequality (13), and the function $Y_{2\alpha}(t)u_0$ is the unique solution of problem (1), (2); consequently, problem (1), (2) is uniformly well posed. The proof of the theorem is complete.

Remark 1. Theorem 1 remains valid for $\alpha = 1$. In this case, the proof is much simpler, and $Y_2(t)u_0 = (1/t)C_1(t)u_0$, where $C_1(t)$ can naturally be referred to as the operator sine function.

Theorem 2. *Let $A \in G_k$, $k > 0$, and let $Y_k(t)$ be the corresponding operator Bessel function. Then the operator A is the generator of an integrated cosine function $C_n(t)$, where n is the least positive integer such that $2n \geq k$.*

Proof. First, note [4] that the operator Bessel function $Y_{2n}(t)$ can be expressed via the operator Bessel function $Y_k(t)$ with the use of the formula for a parameter shift:

$$Y_m(t) = \frac{2}{\mathrm{B}(k/2+1/2, m/2-k/2)} \int_0^1 \left(1-s^2\right)^{(m-k)/2-1} s^k Y_k(ts)ds, \qquad m > k,$$

where $\mathrm{B}(\cdot\,,\cdot)$ is the Euler beta function.

Let $u_0 \in D(A^n)$. By Theorem 3 in [9], the function

$$Y_0(t)u_0 = \frac{t}{(2n-1)!!}\left(\frac{1}{t}\frac{d}{dt}\right)^n \left(t^{2n-1}Y_{2n}(t)u_0\right) \tag{15}$$

is the unique solution of the equation

$$u''(t) = Au(t), \qquad t > 0, \tag{16}$$

with the initial conditions (2).

By using the relation [10, Eq. (1.13)]

$$\left(\frac{1}{t}\frac{d}{dt}\right)^n \left(t^{2n-1}Y_{2n}(t)u_0\right) = \sum_{j=0}^{n} \frac{2^{n-j}C_n^j\Gamma(n+1/2)}{\Gamma(j+1/2)} t^{2j-1} \left(\frac{1}{t}\frac{d}{dt}\right)^j Y_{2n}(t)u_0$$

and the expression [4]

$$Y_k'(t)u_0 = \frac{t}{k+1}Y_{k+2}(t)Au_0$$

for the derivative of the operator Bessel function, we rewrite formula (15) as

$$Y_0(t)u_0 = \sum_{j=0}^{n} \frac{2^{n-j}C_n^j\Gamma(n+1/2)}{\Gamma(j+1/2)} t^{2j} Y_{2n+2j}(t)A^j u_0. \tag{17}$$

This, together with (3), implies the estimate

$$\|Y_0(t)u_0\| \leq M_1 e^{\omega_1 t} \sum_{j=0}^{n} \left\|A^j u_0\right\|, \qquad \omega_1 > \omega.$$

Therefore, problem (16), (2) is exponentially uniformly n-well posed. It follows from Theorem 1.3 in [11] that the operator A is the generator of an integrated cosine function $C_n(t)$. The proof of the theorem is complete.

Remark 2. By virtue of the uniqueness of the solution, we have $C_n(t)u_0 = I^n Y_0(t)u_0$ and $u_0 \in D(A^n)$, where $Y_0(t)$ is given by (15). After integration by parts, we obtain

$$C_n(t)u_0 = I^n Y_0(t)u_0 = \frac{1}{(n-1)!\,(2n-1)!!}\int_0^t (t-s)^{n-1} s \left(\frac{1}{s}\frac{d}{ds}\right)^n \left(s^{2n-1} Y_{2n}(s)u_0\right) ds$$

$$= \frac{(-1)^n}{(n-2)!\,(2n-1)!!}\left(s^{2n-1}\left(\frac{1}{s}\frac{d}{ds}\right)^{n-2}\left(\frac{(t-s)^{n-2}}{s}\right) Y_{2n}(s)u_0\bigg|_0^t - \int_0^t s^{2n}\left(\frac{1}{s}\frac{d}{ds}\right)^{n-1}\left(\frac{(t-s)^{n-2}}{s}\right) Y_{2n}(s)u_0 ds\right)$$

for $n \geq 2$; moreover, the operator function $I^n Y_0(t)$, originally defined on the dense [4] set $D(A^n)$, can be extended to the entire space E. For example,

$$C_1(t) = tY_2(t), \qquad C_2(t) = \frac{t^2}{3}Y_4(t) + \frac{1}{3}\int_0^t sY_4(s)ds,$$

$$C_3(t) = \frac{t^3}{15}Y_6(t) + \frac{t}{5}\int_0^t sY_6(s)ds,$$

and the inverse formulas read

$$Y_2(t) = \frac{1}{t}C_1(t), \qquad Y_4(t) = \frac{3}{t^2}C_2(t) - \frac{3}{t^3}\int_0^t C_2(s)ds,$$

$$Y_6(t) = \frac{15}{t^3}C_3(t) - \frac{45}{t^5}\int_0^t sC_3(s)ds.$$

In conclusion, we recall the definition of integrated semigroup and show how to use it so as to weaken the conditions imposed on the operator A occurring in the problem [12]

$$v'(t) + \frac{k}{t}v(t) = Av(t) + \frac{k}{t}g, \qquad t > 0, \tag{18}$$

$$\lim_{t\to 0}\left(t^k v(t)\right) = v_0. \tag{19}$$

Definition 2. Let $\alpha > 0$. A one-parameter family of bounded linear operators $T_\alpha(t)$, $t \geq 0$, is called an *α-times integrated semigroup* if the following conditions are satisfied:

1. $\Gamma(\alpha)T_\alpha(t)T_\alpha(s) = \int_s^{t+s}(t+s-r)^{\alpha-1}T_\alpha(r)dr - \int_0^t (t+s-r)^{\alpha-1}T_\alpha(r)dr,\ \ t,s \geq 0.$
2. $T_\alpha(0) = 0$.
3. $T_\alpha(t)x$ is a continuous function of $t \geq 0$ for each $x \in E$.
4. There exist constants $M > 0$ and $\omega \in R$ such that $\|T_\alpha(t)\| \leq Me^{\omega t}$, $t \geq 0$.

The generator A of an integrated semigroup $T_\alpha(t)$ is defined as follows: the domain $D(A)$ is the set of elements $x \in E$ such that there exists an element $y \in E$ satisfying the relation

$$T_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x = \int_0^t T_\alpha(s)y\,ds, \qquad t \geq 0; \tag{20}$$

in this case, we set $Ax = y$.

Theorem 3. *Let $k \in N$, let A be the generator of a k-times integrated semigroup $T_k(t)$, and let $g \in D(A)$ and $v_0 \in D\left(A^{k+1}\right)$. Then the function*

$$v(t) = t^{-k}\left(k!\,T_k(t)g + T_k^{(k)}(t)v_0\right)$$

is the unique solution of problem (18), (19), *and moreover,*

$$\|v(t)\| \le Me^{\omega t}\left(\|g\| + \left\|A^k v_0\right\|\right) + \sum_{j=0}^{k} \frac{t^{j-k}}{j!}\left\|A^j v_0\right\|. \tag{21}$$

Proof. Note that the representation of the solution $v(t)$ of problem (18), (19) was obtained from Theorem 7 in [12], where it was assumed that A is the generator of a C_0-semigroup $T(t)$, by replacing the kth fractional integral of the semigroup $T(t)$ by the integrated semigroup $T_k(t)$.

By taking into account the relation (e.g., see [3])

$$T_k^{(k)}(t)v_0 = T_k(t)A^k v_0 + \sum_{j=0}^{k-1} \frac{t^j}{j!}A^j v_0 \tag{22}$$

and definition (20) of the generator of the integrated semigroup $T_k(t)$, we compute $v'(t)$. We obtain

$$v'(t) = -kt^{-k-1}\left(k!\,T_k(t)g + T_k^{(k)}(t)v_0\right) + t^{-k}\left(k!\,AT_k(t)g + kt^{k-1}g + AT_k^{(k)}(t)v_0\right);$$

therefore,

$$v'(t) + \frac{k}{t}v(t) = Av(t) + \frac{k}{t}g;$$

i.e., the function $v(t)$ satisfies Eq. (18).

It also follows from (20) and (22) that the initial condition is satisfied, since

$$\lim_{t\to 0}\left(t^k v(t)\right) = k!\lim_{t\to 0} T_k(t)g + \lim_{t\to 0} T_k^{(k)}(t)v_0 = v_0.$$

The estimate (21) is a consequence of item 4 of Definition 2 and relations (20) and (22). Indeed,

$$\|v(t)\| \le t^{-k}\left(k!\,M_1 t^k e^{\omega t}\|g\| + M_1 t^k e^{\omega t}\left\|A^k v_0\right\| + \sum_{j=0}^{k-1}\frac{t^j}{j!}\left\|A^j v_0\right\|\right)$$
$$= Me^{\omega t}\left(\|g\| + \left\|A^k v_0\right\|\right) + \sum_{j=0}^{k}\frac{t^{j-k}}{j!}\left\|A^j v_0\right\|.$$

Finally, to prove the uniqueness, we note that the change of variables

$$v(t) = t^{-k}w(t) + k!\,t^{-k}T_k(t)g$$

reduces problem (18), (19) to the problem

$$w'(t) = Aw(t), \qquad w(0) = v_0,$$

which, by virtue of Theorem 1.2 in [3], has a unique solution. The proof of the theorem is complete.

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