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MATHEMATICS

To the Solution of the Inverse Sturm–Liouville Problem on the Real Axis

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1. THE MAIN FADDEEV–MARCHENKO THEOREM

Consider the inverse Sturm–Liouville problem

$$-y'' + q(x)y = k^2y, \quad x \in \mathbb{R} \tag{1}$$

on the entire axis in classical setting, where

$$q \in C(\mathbb{R}), \quad \int_{\mathbb{R}}(1+|x|)|q(x)|dx < \infty. \tag{2}$$

Beginning in the middle of the past century, this problem has been considered by many authors (see monographs [1–3], which include an extensive bibliography).

As is well known [2], under condition (2), the discrete spectrum of the initial problem in the class of functions vanishing at infinity is at most finite; all points in the spectrum are simple and lie on the negative half-axis. Taking them into account involves no difficulties both when applying the classical method for solving the inverse problem and when using the approach proposed in this paper. Thus, in what follows, we assume that the coefficient $q(x)$ in Eq. (1) is such that the problem has no discrete spectrum.

Let us describe the classical approach of I.M. Gel'fand, B.M. Levitan, L.D. Faddeev, and V.A. Marchenko to solving this problem. Consider the Jost functions $f_j(x, k)$, $j = 1, 2$, as solutions of the Sturm–Liouville equation with given asymptotic behavior at infinity:

$$f_j(x, k) = e^{\pm ikx} + o(1), \quad f'_j(x, k) = \pm ike^{\pm ikx} + o(1)$$

as $k \to \pm\infty$, where $\pm 1 = (-1)^{j-1}$. They are determined uniquely; in particular, $f_j(x, -k) = \overline{f_j(x, k)}$. For a fixed real number $k \neq 0$, the pairs of functions $\{f_j(x, k), \overline{f_j(x, k)}\}$ form two fundamental systems of solutions and are, therefore, related by

$$f_1(x, k) = b(k)f_2(x, k) + a(k)\overline{f_2(x, k)} \tag{3}$$

with some coefficients $a(k)$ and $b(k)$. The functions f_j can be represented as [2]

$$f_1(x, k) = e^{ikx} + \int_x^{\infty} A_1(x, t)e^{ikt}dt, \qquad f_2(x, k) = e^{-ikx} + \int_{-\infty}^{x} A_2(x, t)e^{-ikt}dt \tag{4}$$

with certain real kernels $A_j(x, t)$; moreover, the integrals converge absolutely. Properties of the coefficients a and b were studied in detail in [2]; we collect them in the in the following proposition.

Lemma 1. *The following relations hold:*

$$a(-k) = \overline{a(k)}, \quad b(-k) = \overline{b(k)},$$

$$|a(k)|^2 = 1 + |b(k)|^2,$$

$$a(k) = 1 + O(k^{-1}), \quad b(k) = O(k^{-1}) \text{ as } k \to \infty,$$

$$\lim_{k\to\infty} k[a(k) + b(k)] = 0.$$

An integral representation of these functions is also known [3]; this is

$$2ika(k) = 2ik - \int_{-\infty}^{\infty} q(t)dt + \int_0^{\infty} A(t)e^{ikt}dt, \qquad 2ikb(k) = \int_{-\infty}^{\infty} B(t)e^{ikt}dt, \tag{5}$$

where $A(t)$ and $B(t)$ are integrable functions, which are expressed in a certain way in terms of the potential q and the kernel $A_j(x, t)$.

According to (4), for each fixed x, the functions $f_j(x, k)$ admit an analytic continuation with respect to

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the parameter k to the upper half-plane $\zeta = k + il$, $l > 0$; for the continued functions, we use the same notation. By virtue of (3), the function a is the Wronskian of f_1 and f_2; therefore, it can be analytically continued to the upper half-plane as well, and the product $\zeta a(\zeta)$ is continuous in the upper half-plane. This can also seen from representation (5). The above assumption that the Sturm–Liouville equation has no spectrum is equivalent to the condition that $a(\zeta) \neq 0$ if $\mathrm{Im}\zeta > 0$. Taking into account Lemma 1, we directly obtain the following property of this function.

Lemma 2. *In the upper half-plane, the function* $a(\zeta)$ *can be represented as*

$$a(\zeta) = \exp\left\{-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\ln(1-|r(t)|^2)}{t-\zeta}dt\right\},$$

$$r(t) = \frac{b(t)}{a(t)},$$

moreover, the product $\zeta a(\zeta)$ *is continuous in the closed upper half-plane.*

The kernels A_j in representation (4) are solutions of the Gel'fand–Levitan–Marchenko integral equations

$$A_1(x, y) + F_1(x+y) + \int_x^{\infty} A_1(x, t)F_1(t+y)dt = 0,$$
$$y \geq x, \tag{6}$$
$$A_2(x, y) + F_2(x+y) + \int_{-\infty}^{x} A_2(x, t)F_2(t+y)dt = 0,$$
$$x \geq y.$$

Under the above assumption about the spectrum, the functions F_j are related to a and b by

$$F_j(x) = \pm\frac{1}{2\pi}\int_{\mathbb{R}}\frac{b(\pm k)}{a(k)}e^{\mp ikx}dk, \quad \pm 1 = (-1)^j. \tag{7}$$

The classical approach to solving the inverse problem under consideration, which was developed by Faddeev, Gel'fand, Levitan, and Marchenko, is as follows.

Main theorem. *Suppose given continuous* (*at* $k \neq 0$) *function* $a(k)$ *and* $b(k)$ *which satisfy the assumptions of Lemmas* 1 *and* 2. *Suppose also that functions* F_j *are continuous and differentiable, and the integrals*

$$\int_x^{\infty}[|F_1(t)| + (1+|t|)|F_1'(t)|]dt,$$

$$\int_{-\infty}^{x}[|F_2(t)| + (1+|t|)|F_2'(t)|]dt$$

are finite for any $x \in \mathbb{R}$.

Then the integral equations (6) *are uniquely solvable with respect to* A_j, *and the functions* $f_j(x, k)$ *determined from the* A_j *according to* (4) *satisfy the Sturm–Liouville equation with coefficient*

$$q(x) = -2[A_1(x, x)]' = 2[A_2(x, x)]'.$$

2. THE REFLECTION COEFFICIENT $r = \dfrac{b}{a}$

Levitan noticed in [4] that the conditions in the main theorem can essentially be reformulated with respect to the function $r(t)$, as follows from Lemma 2. Moreover, the function q can be determined by using a function F_2 whose expression (7) involves only $r(k)$. According to Lemma 1, the function $r(k)$ is strictly less than 1 in absolute value for $k \neq 0$. Moreover, the first two relations in the lemma can be extended to $r(k)$. Thus, we can write

$$r(-k) = \overline{r(k)}, \quad |r(k)| < 1, \quad k \neq 0. \tag{8}$$

In particular, if the limit $r(0) = \lim r(k)$ as $k \longrightarrow 0$ exists, then it is real.

The behavior of $r(k)$ as $k \longrightarrow 0$ is easy to describe by using the integral representations of the functions a and b. Let us write them in the forms $2ika(k) = 2ik + \tilde{a}(k)$ and $2ikb(k) = \tilde{b}(k)$ with the corresponding continuous functions $\tilde{a}$ and $\tilde{b}$. Then, according to Lemma 1, we have $\tilde{a}(0) + \tilde{b}(0) = 0$, and, in the same notation,

$$r(k) = \frac{\tilde{b}(k)}{2ik + \tilde{a}(k)}.$$

If $\tilde{b}(0) \neq 0$, then $\tilde{a}(0)$ is nonzero as well and, hence, the limit $r(0) = \lim r(k)$ as $k \longrightarrow 0$ exists and equals

$$r(0) = \frac{\tilde{b}(0)}{\tilde{a}(0)} = -1.$$

If $\tilde{b}(0) = 0$, then by virtue of the same considerations, we have $\tilde{a}(0) = 0$, and the behavior of $r(k)$ as $k \longrightarrow 0$ may be arbitrary; its character has not been completely clarified so far. Apparently, it is for this reason that, in [2], the coefficients q are subjected to the more restrictive constraint

$$\int_{\mathbb{R}}(1+|x|^2)|q(x)|dx < \infty.$$

Under this constraint, the last condition in Lemma 1 is replaced by $a(k) + b(k) = O(1)$.

The behavior of the function $a(\zeta)$ defined in the upper half-plane $D_+ = \{\mathrm{Im}\zeta > 0\}$ by Lemma 2 was described under natural assumptions about $r(k)$ by Levitan in [4]. His result can be reformulated as follows.

Theorem (Levitan)**.** *Suppose that a function* $r(k) \in C(\mathbb{R})$ *with properties* (8) *satisfies the Hölder condition outside any neighborhood of zero and* $r(k) = O(k^{-1})$ *as* $k \longrightarrow \infty$. *Suppose also that the function* $f_0(k) = \ln[1 -$

$|r(k)|^2] - 2\sigma\ln|k|$, *where* $\sigma = 0$ *if* $|r(0)| < 1$ *and* $\sigma > 0$ *if* $|r(0)| = 1$, *satisfies the Hölder condition in some neighborhood of zero. Then the function*

$$h_0(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\ln[1-|r(t)|^2]dt}{t-\zeta} - \sigma\ln\zeta,$$

$$\operatorname{Im}\zeta > 0$$

is continuous in the closed upper half-plane and satisfies the Höder condition at $\zeta = 0$.

This result can be somewhat strengthened. Let $H(G)$ denote the class of functions satisfying the Hölder condition on a bounded set $G \subseteq \mathbb{C}$. If G is unbounded, then we consider the closure $\hat{G} = \bar{G} \cup \{\infty\}$ of G in the metric of the Riemann sphere. In this case, by $H(\hat{G})$ we understand the class of functions satisfying the Hölder condition with respect to this metric. Equivalently, $\varphi \in H(\hat{G})$ if the functions $\varphi_1(z) = \varphi(z)$ and $\varphi_2(z) = \varphi\left(\frac{1}{z}\right)$ satisfy the Hölder condition on the sets $G_1 = G \cap \{|z| < 2\}$ and $G_2 = \left\{z|\frac{1}{z} \in G\right\} \cap \{|z| < 2\}$, respectively. For $\varphi(\infty) = 0$, we denote the corresponding class by $\mathring{H}(\hat{G})$.

As is known [5], the Cauchy-type integral

$$\phi(z) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\varphi(t)dt}{t-z}, \quad z \in D_{\pm}$$

with density $\varphi \in \mathring{H}(\hat{\mathbb{R}})$ defines a function $\phi \in \mathring{H}(\hat{D}_{\pm})$, and the boundary values $\phi^{\pm}(t_0)$, $t_0 \in \mathbb{R}$, of this function satisfy the Sokhotskii–Plemelj formula

$$2\phi^{\pm}(t_0) = \pm\varphi(t_0) + (S\varphi)(t_0) \tag{9}$$

with singular Cauchy operator

$$(S\varphi)(t_0) = \frac{1}{\pi i}\int_{\mathbb{R}}\frac{\varphi(t)dt}{t-t_0}, \quad t_0 \in \mathbb{R}.$$

In particular, this operator is invariant in the class $\mathring{H}(\hat{\mathbb{R}})$ and has the property $S^2\varphi = \varphi$. If, in addition, its derivative φ' belongs to $\mathring{H}(\hat{\mathbb{R}})$, then, differentiating by parts, we obtain

$$\phi'(z) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\varphi(t)dt}{(t-z)^2} = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\varphi'(t)dt}{t-z} \in \mathring{H}(\hat{D}_{\pm}).$$

Lemma 3. (a) *Suppose that a function* $r(t)$ *is twice differentiable, its derivatives satisfy the condition* $r^{(j)} \in \mathring{H}(\hat{\mathbb{R}})$ *for* $j = 0, 1, 2$ *and* $|r(t)| < 1$ *for* $t \in \mathbb{R}$. *Then the Cauchy-type integral*

$$h(\zeta) = \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\ln[1-|r(t)|^2]dt}{t-\zeta}$$

and its derivatives h' *and* h'' *belong to the class* $\mathring{H}(\hat{D}_+)$.

(b) *Suppose that* $r^{(j)} \in \mathring{H}(\hat{\mathbb{R}})$ *for* $j = 0, 1, 2$, $|r(t)| < 1$ *for* $t \neq 0$, $|r(0)| = 1$, *and, for some* $0 < \sigma \leq 1$, *the function* $f_0(t) = \ln[1 - |r(t)|^2] - 2\sigma\ln|t|$ *and its derivatives* f_0' *and* f_0'' *belong to the class* H *in a neighborhood of zero. Then the function*

$$h_0(\zeta) = h(\zeta) - \sigma\ln\frac{\zeta}{\zeta+i}, \quad \zeta \in D_+,$$

and its derivatives h_0' *and* h_0'' *belong to the class* $\mathring{H}(\hat{D}_+)$.

This lemma implies that, under the assumptions of (a), the function $a(\zeta) = e^{-h(\zeta)}$ is invertible in the class $H(\hat{D})$ together with its derivatives a' and a''. Similarly, under the assumption of (b), this condition is satisfied by the function

$$a_0(\zeta) = \left(\frac{\zeta}{\zeta+i}\right)^{\sigma}a(\zeta).$$

By using Lemma 3, it is easy to determine a class of functions $r(k)$ for which the coefficient a defined by the formula of Lemma 2 and $b = ar$ satisfy all assumptions of the main theorem.

Theorem 1. *Suppose that a function* $r(t) \in C^2(\mathbb{R})$ *satisfies conditions* (8) *and* $(1 + |t|)r^{(j)}(t) \in \mathring{H}(\hat{\mathbb{R}})$ *for* $j = 0, 1, 2$. *Suppose also that* $-1 \leq r(0) < 1$, *and if* $r(0) = -1$, *then the function* $f_0(t) = \ln[1 - |r(t)|^2] - 2\ln|t|$, *together with its first and second derivatives, belongs to the class* H *in a neighborhood of zero. Then the coefficients* $a(t) = e^{-h(t)}$ *and* $b(t) = r(t)e^{-h(t)}$, *where*

$$h(t_0) = \frac{\ln[1-|r(t_0)|^2]}{2} + \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\ln[1-|r(t)|^2]}{t-t_0}dt$$

satisfy all assumptions of the main theorem.

3. A FUNCTION-THEORETIC APPROACH

Under the assumptions of the main theorem, the inverse Sturm–Liouville problem is uniquely solvable, and its solution can be obtained from the Gel'fand–Levitan–Marchenko integral equation. However, there is also a well-known functional-theoretic approach to describing this solution, which is based on the Markushevich problem for the Jost functions.

Suppose that, in accordance with the main theorem, the function $r = \frac{b}{a}$ is continuous and vanishes at infinity, and the analytic function $a(\zeta)$ determined by r satisfies the assumptions of Lemma 2. It is more convenient to pass from the Jost functions $f_j(x, k)$ to the new pair of functions $\phi_j(x, k)$ defined by

$$f_1(x, k) = e^{ikx}[a(k)\phi_1(x, k) + 1],$$
$$f_2(x, k) = e^{-ikx}[\phi_2(x, k) + 1]. \tag{10}$$

By virtue of (4), they can be represented in the form

$$\phi_1(x, k) = a^{-1}(k)\int_x^{\infty} A_1(x, t)e^{ik(t-x)}dt,$$

$$\phi_2(x, k) = \int_{-\infty}^{x} A_2(x, t)e^{ik(x-t)}dt.$$

Obviously, for each fixed x, the functions $\phi_j(x, k)$ admit an analytic continuation with respect to the parameter k to the upper half-plane $\zeta = k + il$, $l > 0$; moreover, they are continuous and bounded in the closed upper half-plane $l \geq 0$ and tend to zero as $l \to \infty$ uniformly in k. Introducing the abbreviated notation

$$\rho(x, k) = r(k)e^{-2ikx},$$
$$h(x, k) = [1 - a^{-1}(k)] + \rho(x, k), \tag{11}$$

we can rewrite the relation (3) between functions (10) in the form

$$\phi_1(x, k) = \rho(x, k)\phi_2(x, k) + \overline{\phi_2(x, k)} + h(x, k),$$
$$k \in \mathbb{R}.$$

Omitting the dependence on x from the notation of ϕ_j and introducing the piecewise-analytic function

$$\phi(\zeta) = \begin{cases} \phi_1(\zeta), & \zeta \in D_+, \\ \overline{\phi_2(\bar{\zeta})}, & \zeta \in D_-, \end{cases} \tag{12}$$

we can regard this relation as a boundary condition in the Markushevich problem [6]:

$$\phi^+ - \phi^- = \rho\overline{\phi^-} + h. \tag{13}$$

Setting $\varphi = \phi^+ - \phi^-$ and taking into account the Sokhotskii–Plemelj formulas (9), we can reduce this problem to the equivalent singular integral equation

$$\varphi - T\varphi = h, \quad T\varphi = \frac{\rho\overline{(-\varphi + S\varphi)}}{2}. \tag{14}$$

This equation was first obtained in well-known paper [7]; however, it has not been studied. Under the assumptions of Theorem 1, it is natural to consider this equation in the class $\mathring{H}(\hat{\mathbb{R}})$. It is easy to show that multiplication by an oscillating coefficient $t \to e^{ixt}$ do not lead beyond this class, so that, according to (11), the function $\rho(t) = \rho(x, t)$ belongs to $\mathring{H}(\hat{\mathbb{R}})$. Recall that $r(t)$ satisfies the conditions $-1 \leq r(0) < 1$ and $|r(t)| < 1$ for $t \neq 0$. According to (11), this condition is also satisfied by the function $\rho(t) = \rho(x, t)$. Therefore, if $\rho(0) \neq -1$, then $|\rho(t)| \leq q < 1$ for all t. In this case, the Markushevich problem (13) is of so-called elliptic type [6], and it is uniquely solvable for the given coefficients [8]. Therefore, so is Eq. (14) in the class $\mathring{H}(\hat{\mathbb{R}})$. As for the case $\rho(0) = -1$, there is a reduction the preceding one.

Theorem 2. *Suppose that a function* $\rho(t) \in \mathring{H}(\hat{\mathbb{R}})$ *is twice continuously differentiable in a neighborhood zero and* $|\rho(t)| < 1$ *for* $t \neq 0$. *Suppose also that, in the case* $|\rho(0)| = 1$, *the condition*

$$(|\rho|)''(0) < 0. \tag{15}$$

holds. Then the singular equation (14) *is uniquely solvable in the class* $\mathring{H}(\hat{\mathbb{R}})$.

Under the assumptions of Theorem 2, not only the operator $1 - T$ but also $1 - T^2$ is invertible. The following lemma shows that T^2 is an integral operator with a weak singularity.

Lemma 4. *For* $\rho \in H(\hat{\mathbb{R}})$, *the operator* T^2 *acts by the formula*

$$(T^2\varphi)(t_0) = \frac{\rho(t_0)}{4\pi i}\int_{\mathbb{R}} \frac{\sigma(t) - \sigma(t_0)}{t - t_0}\varphi(t)dt, \tag{16}$$

where

$$\sigma(t_0) = \overline{\rho(t_0)} - \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\overline{\rho(t)}dt}{t - t_0} \in \mathring{H}(\hat{\mathbb{R}}),$$

and for the operator $1 - T^2$, *the Fredholm alternative holds.*

Theorem 2 and Lemma 4 directly imply the following representation of the Jost functions of the inverse Sturm–Liouville problem.

Theorem 3. *Suppose that functions* $r(t)$ *satisfy the assumptions of Theorem* 1 *and*

$$\sigma(x, t_0) = e^{2ixt_0}\overline{r(t_0)} - \frac{1}{\pi i}\int_{\mathbb{R}} \frac{e^{2ixt}\overline{r(t)}dt}{t - t_0},$$

$$g(x, t_0) = \frac{e^{-2ixt_0}r(t_0)a(t_0) - 2\overline{a(t_0)}}{2|a(t_0)|^2}$$
$$+ \frac{2 + e^{-2ixt_0}r(t_0) - |r(t_0)|^2}{2}$$
$$+ \frac{e^{-2ixt_0}r(t_0)}{2\pi i}\int_{\mathbb{R}} \frac{1 - e^{2ixt}\overline{r(t)a(t)}}{\overline{a(t)}}\frac{dt}{t - t_0}.$$

Then the Fredholm equation

$$\varphi(x, t_0) - \frac{e^{-2ixt_0}r(t_0)}{4\pi i}\int_{\mathbb{R}} \frac{\sigma(x, t) - \sigma(x, t_0)}{t - t_0}\varphi(t)dt = g(x, t_0)$$

is uniquely solvable in the class $\mathring{H}(\mathbb{R})$, *and the Jost functions of the Sturm–Liouville problem are expresses as*

$$f_1(x,k) = e^{ixk}\left[1 + \frac{a(k)\varphi(x,k)}{2} + \frac{a(k)}{2\pi i}\int_{\mathbb{R}}\frac{\varphi(x,t)dt}{t-k}\right],$$

$$f_2(x,k) = e^{-ixk}\left[1 - \frac{\varphi(x,k)}{2} - \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\varphi(x,t)dt}{t-k}\right].$$

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